# Lifting up the proof theory to the countables : Zermelo-Fraenkel set theory *

Toshiyasu Arai †
Graduate School of Science, Chiba University
1-33, Yayoi-cho, Inage-ku, Chiba, 263-8522, JAPAN
tosarai@faculty.chiba-u.jp

**Abstract**

We describe the countable ordinals in terms of iterations of Mostowski collapsings. This gives a proof-theoretic bound on definable countable ordinals in Zermelo-Fraenkel set theory ZF.

## 1 Introduction

In these decades ordinal analyses (mainly of set theories) have progressed greatly, cf. M. Rathjen's contributions [10–13] and [2–4].

Current ordinal analyses are *recursive*. By recursive ordinal analyses we mean that everything in the analyses is recursive (on $\omega$). Namely notation systems for ordinals to measure the proof-theoretic strengths of formal theories are recursive, and operations on (codes of recursive) infinite derivations to eliminate cut inferences are recursive, and so on. Moreover in the analyses we consider only derivations of recursive statements on the least recursively regular ordinal $\omega_1^{CK}$. We now ask: Can we lift up recursive ordinal analyses to countables through a non-effective ordinal analysis? By an analysis on countables we aim at bounding provability in formal theories for sets with respect to statements on countable sets.

The proof technique in these ordinal analyses (cut-elimination with collapsing functions) has been successful in describing the bounds on provability in theories on *recursive analogues* of (small) large cardinals, which were introduced by Richter and Aczel [14]. We can expect that the technique works also for set theories of (true) large cardinals. In this paper we give a way to describe a bound on provability in Zermelo-Fraenkel set theory ZF. We describe the countable ordinal $\Psi_{\omega_1}\varepsilon_{I+1}$, and show that the ordinal is a proof-theoretic bound on definable countable ordinals provably existing in Zermelo-Fraenkel set theory ZF, Theorem 1.1.

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Let us describe the content of this paper. In section 2 we give a characterization of the regularity of ordinals in terms of $\Sigma_1$-Skolem hulls. In section 3 we introduce a theory of sets which is equivalent to $\mathsf{ZF}+(V=L)$, and in section 4 collapsing functions $\alpha \mapsto \Psi_{\kappa,n}\alpha < \kappa$ are introduced for each uncountable regular cardinal $\kappa \leq I$ and $n < \omega$, cf. Definition 4.1, where $I$ is intended to denote the least weakly inaccessible cardinal. Let $\omega_k(I+1)$ denote the tower of $\omega$ with the next epsilon number $\varepsilon_{I+1} = \sup\{\omega_k(I+1) : k < \omega\}$ above $I$. It is easy to see that the predicate $x = \Psi_{\kappa,n}\alpha$ is a $\Sigma_{n+1}$-predicate for $\alpha < \varepsilon_{I+1}$, and for each $n, k < \omega$ $\mathsf{ZF}+(V=L)$ proves $\forall\alpha < \omega_k(I+1)\forall\kappa \leq I\exists x < \kappa[x = \Psi_{\kappa,n}\alpha]$, cf. Lemma 4.6.

Conversely we show the following Theorem 1.1 in the fragment $I\Sigma^0_1$ of first-order arithmetic.

**Theorem 1.1** *For a sentence $\exists x \in L_{\omega_1}\, \varphi(x)$ with a first-order formula $\varphi(x)$, if*

$$\mathsf{ZF}+(V=L) \vdash \exists x \in L_{\omega_1}\, \varphi(x)$$

*then*

$$\exists n < \omega[\mathsf{ZF}+(V=L) \vdash \exists x \in L_{\Psi_{\omega_1,n}\omega_n(I+1)}\varphi(x)].$$

**Remark**. From Theorem 1.1 together with Lemma 4.6 it follows that the countable ordinal

$$\Psi_{\omega_1}\varepsilon_{I+1} := \sup\{\Psi_{\omega_1,n}\omega_n(I+1) : n < \omega\}$$

is the limit of $\mathsf{ZF}+(V=L)$-provably countable ordinals in the following sense:

$$\Psi_{\omega_1}\varepsilon_{I+1} = \sup\{\alpha < \omega_1 : \alpha \text{ is a } \mathsf{ZF}+(V=L)\text{-provably countable ordinal }\}$$

where by saying that an ordinal $\alpha$ is a $\mathsf{ZF}+(V=L)$-provably countable we mean

$$\mathsf{ZF}+(V=L) \vdash \exists! x < \omega_1\, \varphi(x) \,\&\, L \models \varphi(\alpha) \text{ for some formula } \varphi.$$

From Theorem 1.1 we see that if $\mathsf{ZF}+(V=L)$ proves the existence of a real $a \in {}^{\omega}\omega$ enjoying a first-order condition $\varphi(a)$, $\mathsf{ZF}+(V=L) \vdash \exists a \in {}^{\omega}\omega\varphi(a)$, then such a real $a$ is already in level $L_{\Psi_{\omega_1}\varepsilon_{I+1}}$ of constructible hierarchy.

This paper is based on a technique, *operator controlled derivations*, which was introduced by W. Buchholz [6], hereby he gave a convincing ordinal analysis for the theory KP$i$ of recursively inaccessible ordinals, which is a recursive analogue of $\mathsf{ZF}$. In section 5 operator controlled derivations for $\mathsf{ZF}$ are introduced, and in the final section 6 Theorem 1.1 is concluded. First let us explain the technique briefly.

In an operator controlled derivation, ordinals occurring in the derivation are controlled by an operator $\mathcal{H}$ on ordinals. Through this we see that these ordinals are in a Skolem hull $\mathcal{H}$. On the other side a recursive notation system is defined through an iteration of Skolem hullings. Suppose that a formal theory on sets

proves a sentence $\exists x < \omega_1^{CK}\theta$ for a bounded formula $\theta$. Then the technique tells us how many times do we iterate Skolem hulllings to bound a recursive ordinal $x$, a witness for $\theta$.

To be specific, let us explain how a Skolem hull looks like. Let $\mathcal{F}$ denote a set of functions.

**Definition 1.2** (Cf. [8].) *For sets $X$, $Cl(X;\mathcal{F})$ denotes the* Skolem hull *of $X$ under the functions in $\mathcal{F}$.*

The set $Cl(X;\mathcal{F})$ is inductively generated as follows.

1. $X \subset Cl(X;\mathcal{F})$.
2. If $\vec{x} \subset Cl(X;\mathcal{F})$, $f \in \mathcal{F}$ and $\vec{x} \subset dom(f)$, then $f(\vec{x}) \in Cl(X;\mathcal{F})$.

Now let us restrict the construction on the class of ordinals $Ord$. $\Omega = \omega_1$ denotes the least uncountable ordinal. Let $\mathcal{F}$ be a *countable* set of ordinal functions $f : Ord^n \to Ord$, where the arity $n < \omega$ of the function $f$ is fixed for each $f$. Assume that 0-ary functions $0, \Omega$ belong to $\mathcal{F}$.

**Proposition 1.3** 
1. *$\forall\alpha < \Omega\exists\beta < \Omega[Cl(\alpha;\mathcal{F}) \cap \Omega \subset \beta]$.*
2. *$\forall\alpha < \Omega\exists\beta < \Omega[\beta > \alpha \,\&\, Cl(\beta;\mathcal{F}) \cap \Omega \subset \beta]$. Namely $\{\beta < \Omega : Cl(\beta;\mathcal{F}) \cap \Omega \subset \beta\}$ is unbounded in $\Omega$.*
3. *$\{\beta < \Omega : Cl(\beta;\mathcal{F}) \cap \Omega \subset \beta\}$ is closed in $\Omega$.*

**Proof**. 1.3.1. If $\alpha < \Omega$, then the set $Cl(\alpha;\mathcal{F})$ is countable.
1.3.2. Given $\alpha < \Omega$, define $\{\beta_n\}_n$ inductively, $\beta_0 = \alpha + 1$, $\beta_{n+1} = \min\{\beta < \Omega : Cl(\beta_n;\mathcal{F}) \cap \Omega \subset \beta\}$. Then $\beta = \sup_n \beta_n$ is a desired one. $\beta < \Omega$ since $\Omega$ is regular. $\Box$
Let us enumerate the closed points. Define sets $Cl_\alpha(X;\mathcal{F})$ and ordinals $\psi_\Omega(\alpha;\mathcal{F})$ by simultaneous recursion on ordinals $\alpha$ as follows.

Let
$$Cl_\alpha(X;\mathcal{F}) := Cl(X;\mathcal{F} \cup \{\psi_\Omega(\cdot;\mathcal{F})\restriction\alpha\})$$
where
$$\psi_\Omega(\alpha;\mathcal{F}) = \min\{\beta \leq \Omega : Cl_\alpha(\beta;\mathcal{F}) \cap \Omega \subset \beta\}.$$
Then a transfinite induction on $\alpha$ shows with Proposition 1.3.2
$$\forall\alpha\exists\beta < \Omega[\psi_\Omega(\alpha;\mathcal{F}) = \beta].$$

For $\mathcal{F}_0 = \{0, \Omega\} \cup \{\lambda xy.x + y, \lambda x.\omega^x\}$(and the Veblen function $\lambda xy.\varphi xy$), $\psi_\Omega(\varepsilon_{\Omega+1};\mathcal{F}_0)$ is the Howard ordinal, the proof-theoretic ordinal of the theory $\mathrm{ID}_1$ for non-iterated positive elementary inductive definition on $\omega$, or equivalently of KP$\omega$, i.e., Kripke-Platek set theory with the axiom of infinity.

Observe that each function in $\mathcal{F}_0$ is $\{\Omega\}$-recursive in $L_\sigma$ for any $\sigma > \Omega$. Here an $\{\Omega\}$-recursive function is $\Sigma$-definable from the 0-ary function $\Omega$, a parameter.

Now let us extend $\mathcal{F}_0$ to the set $\mathcal{F}_{all}$ of *all* $\{\Omega\}$-recursive functions on $L_\sigma$. Then it turns out that $Cl(\alpha;\mathcal{F}_{all})$ is the $\Sigma_1$-Skolem hull $\mathrm{Hull}^\sigma_{\Sigma_1}(\alpha \cup \{\Omega\})$ of $\alpha \cup \{\Omega\}$ on $L_\sigma$, and this gives a characterization of the regularity of the ordinal $\Omega$, cf. Theorem 2.10 below.

## 2 $\Sigma_n$-Skolem hulls

For a model $\langle M; \in\restriction(M \times M)\rangle$ and $X \subset M$, $\Sigma_n^M(X)$ denotes the set of $\Sigma_n(X)$-definable subsets of $M$, where $\Sigma_n(X)$-formulae may have parameters from $X$. $\Sigma_n^M(M)$ is denoted $\Sigma_n(M)$.

An ordinal $\alpha > 1$ is said to be a *multiplicative principal number* iff $\alpha$ is closed under ordinal multiplication, i.e., $\exists\beta[\alpha = \omega^{\omega^\beta}]$. If $\alpha$ is a multiplicative principal number, then $\alpha$ is closed under Gödel's pairing function $j$ and there exists a $\Delta_1$-bijection between $\alpha$ and $L_\alpha$ for the constructible hierarchy $L_\alpha$ up to $\alpha$. In this section $\sigma$ is assumed to be a multiplicative principal number$> \omega$.

**Definition 2.1** *1. Reg* denotes the class of uncountable regular ordinals.

2. $cf(\kappa) := \min\{\alpha \leq \kappa : \text{there is a cofinal map } f : \alpha \to \kappa\}$.

$$\begin{array}{rcl} \kappa \text{ is } \textit{uncountable regular} & :\Leftrightarrow & \kappa \in Reg \Leftrightarrow \kappa > \omega \,\&\, cf(\kappa) = \kappa \\ & \Leftrightarrow & \kappa > \omega \,\&\, \forall\alpha < \kappa(\alpha < cf(\kappa)) \end{array}$$

$card(\alpha) < card(\kappa) :\Leftrightarrow$ there is no surjective map $f : \alpha \to \kappa$.

3. $\rho(L_\sigma)$ denotes the $\Sigma_1$*-projectum* of $L_\sigma$: $\rho(L_\sigma)$ is the least ordinal $\rho$ such that $\mathcal{P}(\rho) \cap \Sigma_1(L_\sigma) \not\subset L_\sigma$.

4. Let $\alpha \leq \beta$ and $f : L_\alpha \to L_\beta$. Then the map $f$ is a $\Sigma_n$*-elementary embedding*, denoted $f : L_\alpha \prec_{\Sigma_n} L_\beta$ iff for any $\Sigma_n(L_\alpha)$-sentence $\varphi[\bar{a}]$ $(\bar{a} \subset L_\alpha)$, $L_\alpha \models \varphi[\bar{a}] \Leftrightarrow L_\beta \models \varphi[f(\bar{a})]$ where $f(\bar{a}) = f(a_1), \ldots, f(a_k)$ for $\bar{a} = a_1, \ldots, a_k$. An ordinal $\gamma$ such that $\forall\delta < \gamma[f(\delta) = \delta] \,\&\, f(\gamma) > \gamma$ is said to be the *critical point* of the $\Sigma_n$- elementary embedding $f$ if such an ordinal $\gamma$ exists.

5. For $X \subset L_\sigma$, $\mathrm{Hull}^\sigma_{\Sigma_n}(X)$ denotes the set ($\Sigma_n$*-Skolem hull* of $X$ in $L_\sigma$) defined as follows. $<_L$ denotes a $\Delta_1$-well ordering of the constructible universe $L$. Let $\{\varphi_i : i \in \omega\}$ denote an enumeration of $\Sigma_n$-formulae in the language $\{\in\}$. Each is of the form $\varphi_i \equiv \exists y\theta_i(x, y; u)\, (\theta \in \Pi_{n-1})$ with fixed variables $x, y, u$. Set for $b \in X$

$$\begin{array}{rcl} r^\sigma_{\Sigma_n}(i, b) & \simeq & \text{the } <_L \text{-least } c \in L_\sigma \text{ such that } L_\sigma \models \theta_i((c)_0, (c)_1; b) \\ h^\sigma_{\Sigma_n}(i, b) & \simeq & (r^\sigma_{\Sigma_n}(i, b))_0 \\ \mathrm{Hull}^\sigma_{\Sigma_n}(X) & = & rng(h^\sigma_{\Sigma_n}) = \{h^\sigma_{\Sigma_n}(i, b) \in L_\sigma : i \in \omega, b \in X\} \end{array} \tag{1}$$

Then $L_\sigma \models \exists x\exists y\, \theta_i(x, y; b) \to h^\sigma_{\Sigma_n}(i, b)\downarrow \,\&\, \exists y\, \theta_i(h^\sigma_{\Sigma_n}(i, b), y; b)$.

The following Propositions 2.2, 2.3 and 2.4 are easy to see.

**Proposition 2.2** *For $a, \kappa \in L_\sigma$, $\mathrm{Hull}^\sigma_{\Sigma_1}(a \cup \{\kappa\}) = Cl(a; \mathcal{F}_{all})$, where in the RHS, Definition 1.2, $\Omega$ is replaced by $\kappa \in \mathcal{F}_{all}$, and $\mathcal{F}_{all}$ denotes the set of all $\{\kappa\}$-recursive (partial) functions on $L_\sigma$. Namely $f \in \mathcal{F}_{all}$ iff there exists an $i < \omega$ such that $f(b) \simeq \beta \Leftrightarrow h^\sigma_{\Sigma_1}(i, \langle b, \kappa\rangle) \simeq \beta$ for $b < \sigma$, where $\langle b, c\rangle$ denotes the pairing of $b$ and $c$.*

**Proposition 2.3** *Assume that $X$ is a set in $L_\sigma$. Then $r^\sigma_{\Sigma_n}$ and $h^\sigma_{\Sigma_n}$ are partial $\Delta_n(L_\sigma)$-maps such that the domain of $h^\sigma_{\Sigma_n}$ is a $\Sigma_n(L_\sigma)$-subset of $\omega \times X$. Therefore its range $\mathrm{Hull}^\sigma_{\Sigma_n}(X)$ is a $\Sigma_n(L_\sigma)$-subset of $L_\sigma$.*

**Proposition 2.4** *Let $Y = \mathrm{Hull}^\sigma_{\Sigma_n}(X)$. For any $\Sigma_n(Y)$-sentence $\varphi(\bar{a})$ with parameters $\bar{a}$ from $Y$ $L_\sigma \models \varphi(\bar{a}) \Leftrightarrow Y \models \varphi(\bar{a})$. Namely $Y \prec_{\Sigma_n} L_\sigma$.*

**Definition 2.5** (Mostowski collapsing function $F$)

Let $n \geq 1$. By Proposition 2.4 and the Condensation Lemma, cf. [9], we have an isomorphism (Mostowski collapsing function)

$$F : \mathrm{Hull}^\sigma_{\Sigma_n}(X) \leftrightarrow L_\gamma$$

for an ordinal $\gamma \leq \sigma$ such that $F \restriction Y = id \restriction Y$ for any transitive $Y \subset \mathrm{Hull}^\sigma_{\Sigma_n}(X)$.

Let us denote, though $\sigma \not\in dom(F) = \mathrm{Hull}^\sigma_{\Sigma_n}(X)$

$$F(\sigma) := \gamma.$$

Also for the above Mostowski collapsing map $F$ let

$$F^{\Sigma_n}(x; \sigma, X) := F(x).$$

The inverse $G := F^{-1}$ of $F$ is a $\Sigma_n$-elementary embedding from $L_{F(\sigma)}$ to $L_\sigma$.

**Definition 2.6** Let $\kappa$ be an ordinal such that $\omega < \kappa < \sigma$, and let

$$F^{\Sigma_n}_{\beta\cup\{\kappa\}}(x) := F^{\Sigma_n}(x; \sigma, \beta \cup \{\kappa\}).$$

Then put

$$\begin{array}{rcl} C^\sigma_{\Sigma_1}(\kappa) & := & \{x < \kappa : x \in Cr^\sigma_{\Sigma_1}(\{\kappa\}) \,\&\, F^{\Sigma_1}_{x\cup\{\kappa\}}(\sigma) < \kappa\} \\ x \in Cr^\sigma_{\Sigma_1}(\{\kappa\}) & :\Leftrightarrow & \mathrm{Hull}^\sigma_{\Sigma_1}(x \cup \{\kappa\}) \cap \kappa \subset x \end{array}$$

**Proposition 2.7** *Let $\alpha$ be a multiplicative principal number with $\omega \leq \alpha < \kappa < \sigma$. Assume that $\sigma$ is recursively regular and the $\Sigma_1$-projectum $\rho(L_\sigma) > \alpha$.*

*Then for the map $h^\sigma_{\Sigma_1}$ with $X = \alpha \cup \{\kappa\}$ in (1) we have $dom(h^\sigma_{\Sigma_1}) \in L_\sigma$. Therefore $\mathrm{Hull}^\sigma_{\Sigma_1}(\alpha \cup \{\kappa\}) = rng(h^\sigma_{\Sigma_1})$ is a set in $L_\sigma$, and the Mostowski collapsing function $F^{\Sigma_1}_{\alpha\cup\{\kappa\}} : \mathrm{Hull}^\sigma_{\Sigma_1}(\alpha \cup \{\kappa\}) \leftrightarrow L_{F^{\Sigma_1}_{\alpha\cup\{\kappa\}}(\sigma)}$ is a $\Delta_1(L_\sigma)$-map. Hence $L_{F^{\Sigma_1}_{\alpha\cup\{\kappa\}}(\sigma)} = rng(F^{\Sigma_1}_{\alpha\cup\{\kappa\}}) \in L_\sigma$, i.e., $F^{\Sigma_1}_{\alpha\cup\{\kappa\}}(\sigma) < \sigma$.*

*Moreover if $\rho(L_\sigma) > \kappa$, then $Cr^\sigma_{\Sigma_1}(\{\kappa\}) = \{x < \kappa : \mathrm{Hull}^\sigma_{\Sigma_1}(x \cup \{\kappa\}) \cap \kappa \subset x\}$ is a set in $L_\sigma$.*

**Proof**. By the definition $dom(h^\sigma_{\Sigma_1}) = \{(i, \beta) \in \omega \times \alpha : L_\sigma \models \exists c\, \theta_i((c)_0, (c)_1; \beta, \kappa)\}$ is a $\Sigma_1(L_\sigma)$-subset of $\omega \times \alpha \leftrightarrow \alpha$.

By the supposition we have $\alpha < \rho(L_\sigma)$. Therefore any $\Sigma_1(L_\sigma)$-subset of $\alpha$ is a set in $L_\sigma$ by the definition of the $\Sigma_1$-projectum.

$Cr^\sigma_{\Sigma_1}(\{\kappa\})$ is a $\Pi_1(L_\sigma)$-subset of $\kappa < \rho(L_\sigma)$, and hence is a set in $L_\sigma$. $\Box$

**Lemma 2.8** *Let $\alpha$ be a multiplicative principal number with $\omega \leq \alpha < \kappa < \sigma$. Assume that $\sigma$ is recursively regular and $L_\sigma \models \alpha < cf(\kappa)$.*

1. *$\alpha < \rho(L_\sigma)$.*
2. *$F^{\Sigma_1}_{\alpha\cup\{\kappa\}}(\sigma) < \kappa$.*
3. *Let $\beta$ denote the least ordinal $\beta \leq \kappa$ such that $\mathrm{Hull}^\sigma_{\Sigma_1}(\alpha \cup \{\kappa\}) \cap \kappa \subset \beta$. Then $\beta < \kappa$ and $L_\sigma \models \beta < cf(\kappa)$, and hence $\beta < \rho(L_\sigma)$.*

**Proof**.
2.8.1(Cf. [5].). Let $\emptyset \neq B \in \Sigma_1(L_\sigma) \cap \mathcal{P}(\alpha)$. We show $B \in L_\sigma$. Let $g : \sigma \to B$ be a surjection, and $f$ be the map $f(\gamma) = g(\mu\delta(g(\delta) \notin \{f(\xi) : \xi < \gamma\}))$, i.e., $f(\gamma)$ is the $\gamma$th member of $B$. Both $g$ and $f$ are $\Delta_1(L_\sigma)$-maps. Suppose that $f$ is total. The $\Sigma_1(L_\sigma)$-injection $f$ from $\sigma$ to $\alpha$ yields an injection from $\kappa$ to $\alpha$ in $L_\sigma$, whose inverse would be a cofinal map from $\alpha$ to $\kappa$ in $L_\sigma$. Let $\gamma_0$ be the least $\gamma < \sigma$ such that $f(\gamma)$ is undefined. Then $B = \{f(\gamma) : \gamma < \gamma_0\}$, and hence $B \in L_\sigma$ by $\Sigma$-Replacement.
2.8.2. We have $\alpha < \rho(L_\sigma)$ by Lemma 2.8.1. Then by Proposition 2.7 we have $F^{\Sigma_1}_{\alpha\cup\{\kappa\}}(\sigma) < \kappa$.
2.8.3. By Proposition 2.7, there exists a surjective map in $L_\sigma$ from $\alpha$ to $\mathrm{Hull}^\sigma_{\Sigma_1}(\alpha \cup \{\kappa\})$. Therefore $\mathrm{Hull}^\sigma_{\Sigma_1}(\alpha \cup \{\kappa\}) \cap \kappa$ is bounded in $\kappa$. By the minimality of $\beta$, $\mathrm{Hull}^\sigma_{\Sigma_1}(\alpha \cup \{\kappa\}) \cap \kappa$ is cofinal in $\beta$. $\Box$

**Proposition 2.9** *Let $n \geq 1$ and $L_\sigma \models$ KP$\omega$ + $\Sigma_n$-Collection. Then for $\kappa \leq \sigma$, $\{(x, y) : x < \kappa \,\&\, y = \min\{y < \kappa : \mathrm{Hull}^\sigma_{\Sigma_n}(x \cup \{\kappa\}) \cap \kappa \subset y\}\}$ is a $Bool(\Sigma_n(L_\sigma))$-predicate on $\kappa$, and hence a set in $L_\sigma$ if $\kappa < \sigma$ and $L_\sigma \models \Sigma_n$-Separation.*

**Proof**. Let $\varphi(y, \kappa)$ be the $\Pi_n$-predicate $\varphi(y, \kappa) :\Leftrightarrow \forall z < \kappa[z \in \mathrm{Hull}^\sigma_{\Sigma_n}(x \cup \{\kappa\}) \to z \in y]$. Then $y = \min\{y < \kappa : \mathrm{Hull}^\sigma_{\Sigma_n}(x \cup \{\kappa\}) \cap \kappa \subset y\}$ iff $y < \kappa \wedge \varphi(y, \kappa) \wedge \forall u < y \neg\varphi(u, \kappa)$, which is $Bool(\Sigma_n(L_\sigma))$ by $\Pi_{n-1}$-Collection. $\Box$

## 2.1 Regularity

$F^{\Sigma_1}_{x\cup\{\kappa\}}(y)$ denotes the Mostowski collapse $F^{\Sigma_1}(y; \sigma, x \cup \{\kappa\})$. The following Theorems 2.10 and 2.12 should be folklore.

**Theorem 2.10** (Cf. [1].) *Let $\sigma$ be an ordinal such that $L_\sigma \models$ KP$\omega$+$\Sigma_1$-Separation, and $\omega \leq \alpha < \kappa < \sigma$ with $\alpha$ a multiplicative principal number and $\kappa$ a limit ordinal. Then the following conditions are mutually equivalent:*

1.
$$L_\sigma \models {}^\alpha\kappa \subset L_\kappa \tag{2}$$

2.
$$L_\sigma \models \alpha < cf(\kappa) \tag{3}$$

3. *There exists an ordinal $x$ such that $x \in C^{\sigma}_{\Sigma_1}(\kappa) \cap (\alpha, \kappa)$, i.e.,*

$$x \in Cr^{\sigma}_{\Sigma_1}(\{\kappa\}) \cap (\alpha, \kappa) \,\&\, F^{\Sigma_1}_{x \cup \{\kappa\}}(\sigma) < \kappa \tag{4}$$

4. *For the Mostowski collapse $F^{\Sigma_1}_{x \cup \{\kappa\}}(y)$*

$$\begin{aligned}&\exists x[\alpha < x = F^{\Sigma_1}_{x \cup \{\kappa\}}(\kappa) < F^{\Sigma_1}_{x \cup \{\kappa\}}(\sigma) < \kappa \,\&\, \forall \Sigma_1\, \varphi \forall a \in L_x \\ &(L_\sigma \models \varphi[\kappa, a] \to L_{F^{\Sigma_1}_{x \cup \{\kappa\}}(\sigma)} \models \varphi[x, a])]\end{aligned} \tag{5}$$

**Proof**. Obviously under the assumption that $\sigma$ is recursively regular, (2) and (3) are mutually equivalent, and (4) implies (5).

Assume $\sigma$ is recursively regular, $\kappa$ denotes a limit ordinal and $\alpha$ a multiplicative principal number with $\omega \leq \alpha < \kappa < \sigma$.

(5) $\Rightarrow$ (2). Suppose there exist an ordinal $x$ such that $\alpha < x = F^{\Sigma_1}_{x \cup \{\kappa\}}(\kappa) < F^{\Sigma_1}_{x \cup \{\kappa\}}(\sigma) < \kappa$ and for any $\Sigma_1$ $\varphi$ and any $a \in L_x$

$$L_\sigma \models \varphi[\kappa, a] \Rightarrow L_{F^{\Sigma_1}_{x \cup \{\kappa\}}(\sigma)} \models \varphi[x, a] \tag{5}$$

Let us show
$$L_\sigma \models {}^\alpha\kappa \subset L_\kappa.$$
Define a $\Delta_1(L_\sigma)$-partial map $S : dom(S) \to {}^\alpha\kappa \cap L_\kappa$ ($dom(S) \subset \kappa$) by letting $S_\beta$ be the $<_L$ least $X \in {}^\alpha\kappa \cap L_\kappa$ such that $\forall \gamma < \beta (X \neq S_\gamma)$.

It suffices to show that $L_\sigma \models {}^\alpha\kappa \subset \{S_\beta\}_\beta = rng(S)$. Suppose there exists an $f \in {}^\alpha\kappa \cap L_\sigma$ so that $\forall \beta < \kappa (S_\beta \neq f)$ and let $f_0$ denote the $<_L$-least such function. Then $f_0$ is $\Sigma_1$ definable on $L_\sigma$ from $\{\alpha, \kappa\}$: for the $\Delta_1(L_\sigma)$-formula $\varphi(f, \alpha, \kappa) :\Leftrightarrow \theta(f, \alpha, \kappa) \land \forall g <_L f \neg\theta(g, \alpha, \kappa)$ with $\theta(f, \alpha, \kappa) :\Leftrightarrow f \in {}^\alpha\kappa \land \forall \beta < \kappa (S_\beta \neq f)$ we have $L_\sigma \models \varphi(f_0, \alpha, \kappa) \,\&\, L_\sigma \models \exists! f \varphi(f, \alpha, \kappa)$. By (5) we have $L_{F^{\Sigma_1}_{x \cup \{\kappa\}}(\sigma)} \models \exists f \varphi(f, \alpha, x)$, i.e., there exists the $<_L$-least $f_1 \in {}^\alpha x \cap L_\kappa$ ($F^{\Sigma_1}_{x \cup \{\kappa\}}(\sigma) \leq \kappa$) such that $\forall \beta < x (< \kappa)(S_\beta \neq f_1)$.

We show $L_\kappa \ni f_1 = f_0$. This yields a contradiction. It suffices to see $f_1 \subset f_0$ for $f_1 : \alpha \to x$ and $f_0 : \alpha \to \kappa$. By (5) we have for $\beta < \alpha$, $\gamma < x$

$$\begin{aligned}&f_1(\beta) = \gamma \Leftrightarrow L_{F^{\Sigma_1}_{x \cup \{\kappa\}}(\sigma)} \models \forall f[\varphi(f, \alpha, x) \to f(\beta) = \gamma] \Rightarrow \\ &L_\sigma \models \forall f[\varphi(f, \alpha, \kappa) \to f(\beta) = \gamma] \Leftrightarrow f_0(\beta) = \gamma\end{aligned}$$

Note that in this proof it suffices to assume that $\sigma$ is recursively regular, and we see that the condition $F^{\Sigma_1}_{x \cup \{\kappa\}}(\sigma) < \kappa$ can be weakened to $F^{\Sigma_1}_{x \cup \{\kappa\}}(\sigma) \leq \kappa$ in (4) and (5).

(3) $\Rightarrow$ (4). Assume $L_\sigma \models \Sigma_1$-Separation, and $L_\sigma \models \alpha < cf(\kappa)$.

We show the existence of an ordinal $x < \kappa$ such that

$$x > \alpha \,\&\, \mathrm{Hull}^{\sigma}_{\Sigma_1}(x \cup \{\kappa\}) \cap \kappa \subset x \,\&\, F^{\Sigma_1}_{x \cup \{\kappa\}}(\sigma) < \kappa.$$

Then $F^{\Sigma_1}_{x\cup\{\kappa\}}(\kappa)=x$.

As in the proof of Proposition 1.3.2, define recursively ordinals $\{x_n\}_n$ as follows. $x_0=\alpha+1$, and $x_{n+1}$ is defined to be the least ordinal $x_{n+1}\leq\kappa$ such that $\mathrm{Hull}^{\sigma}_{\Sigma_1}(x_n\cup\{\kappa\})\cap\kappa\subset x_{n+1}$. We see inductively that $x_n<\kappa$ from Lemma 2.8.3. On the other hand we have ${}^n\kappa\subset L_\kappa$ by (2). Moreover by Proposition 2.9, the map $n\mapsto x_n$ is a $\Delta_1$-set in $L_\sigma\models\Sigma_1$-Separation.

Therefore $x=\sup_n x_n<\kappa$ enjoys $x>\alpha$, and $\mathrm{Hull}^{\sigma}_{\Sigma_1}(x\cup\{\kappa\})\cap\kappa\subset x$.

It remains to see $F^{\Sigma_1}_{x\cup\{\kappa\}}(\sigma)<\kappa$. By Lemma 2.8.2 it suffices to see $x<cf(\kappa)$.

Since there exists a $\Delta_1(L_\sigma)$-surjective map $h_n:x_n\to\mathrm{Hull}^{\sigma}_{\Sigma_1}(x_n\cup\{\kappa\})$, pick an increasing cofinal map $f_n:x_n\to x_{n+1}$ in $L_\sigma$ using the minimality of $x_{n+1}$. Using the uniformity of $f_n$, we see the existence of an increasing cofinal map $f:\alpha\to x$ in $L_\sigma$. Therefore $L_\sigma\models x<cf(\kappa)$. □

**Remark**. In the proof of Theorem 2.10, the assumption that $L_\sigma\models\Sigma_1$-Separation is used only in the part (3) ⇒ (4), and everything except the part holds when $\sigma$ is recursively regular.

**Corollary 2.11** *Suppose $\kappa$ is uncountable regular in $L_\sigma\models\mathrm{KP}\omega{+}\Sigma_1$-Separation.*

1. *$\kappa$ is $\sigma$-stable, i.e., $L_\kappa\prec_{\Sigma_1}L_\sigma$.*
2. $\{\lambda<\kappa:\lambda\in Reg\}=\{\lambda<\kappa:\lambda$ is uncountable regular in $L_\sigma\}$ *is a $\Delta_0$-subset of $\kappa$. Therefore the map $\kappa>\alpha\mapsto\omega_\alpha$ is a $\Delta_1$-map on $L_\sigma$. On the other side the map $\sigma>\alpha\mapsto\omega_\alpha$ is a $\Delta_2$-map on $L_\sigma$.*

**Proof**. 2.11.1. Let $\varphi[a]$ be a $\Sigma_1$-formula with a parameter $a\in L_\kappa$. Pick an $\alpha_a\in C^{\sigma}_{\Sigma_1}(\kappa)$ such that $a\in L_{\alpha_a}$ by Theorem 2.10. Then $L_\sigma\models\varphi[a]\Rightarrow L_{F^{\Sigma_1}_{\alpha_a\cup\{\kappa\}}}\models\varphi[a]\Rightarrow L_\kappa\models\varphi[a]$ for $a=F^{\Sigma_1}_{\alpha_a\cup\{\kappa\}}(a)$ and $F^{\Sigma_1}_{\alpha_a\cup\{\kappa\}}(\sigma)<\kappa$.
2.11.2. For $\lambda<\kappa$, we see from Corollary 2.11.1, $L_\sigma\models\lambda\in Reg\Leftrightarrow L_\kappa\models\lambda\in Reg$. □

For the existence of power sets we have the following Theorem 2.12.

**Theorem 2.12** (Cf. [1].) *Let $\sigma$ be recursively regular, and $\omega\leq\alpha<\kappa<\sigma$ with $\alpha$ a multiplicative principal number and $\kappa$ a limit ordinal. Then the following conditions are mutually equivalent:*

1. $$\alpha<\rho(L_\sigma)\land F^{\Sigma_1}_{\alpha\cup\{\alpha,\kappa\}}(\sigma)=F^{\Sigma_1}(\sigma;\sigma,\alpha\cup\{\alpha,\kappa\})<\kappa \tag{6}$$
2. *For the Mostowski collapse* $F^{\Sigma_1}_{\alpha\cup\{\alpha,\kappa\}}:\mathrm{Hull}^{\sigma}_{\Sigma_1}(\alpha\cup\{\alpha,\kappa\})\leftrightarrow L_{F^{\Sigma_1}_{\alpha\cup\{\alpha,\kappa\}}(\sigma)}$
$$\exists x[\alpha<x\leq F^{\Sigma_1}_{\alpha\cup\{\alpha,\kappa\}}(\kappa)<F^{\Sigma_1}_{\alpha\cup\{\alpha,\kappa\}}(\sigma)<\kappa\,\&\,\forall\Sigma_1\,\varphi\forall a\in L_x$$
$$(L_\sigma\models\varphi[\kappa,a]\to L_{F^{\Sigma_1}_{\alpha\cup\{\alpha,\kappa\}}(\sigma)}\models\varphi[F^{\Sigma_1}_{\alpha\cup\{\alpha,\kappa\}}(\kappa),a])] \tag{7}$$

*3.*

$$\mathcal{P}(\alpha) \cap L_\sigma \subset L_\kappa \tag{8}$$

*4.*

$$L_\sigma \models card(\alpha) < card(\kappa) \tag{9}$$

**Proof**. In showing the direction (6)⇒(7), pick the least ordinal $x > \alpha$ not in $\mathrm{Hull}^{\sigma}_{\Sigma_1}(\alpha \cup \{\alpha, \kappa\})$. (8)⇒(9) and (9)⇒(6) are easily seen.

(7)⇒(8). As in the proof of (5) ⇒ (2), define a $\Delta_1$-partial map $S : dom(S) \to \mathcal{P}(\alpha) \cap L_\kappa$ ($dom(S) \subset \kappa$) by letting $S_\beta$ be the $<_L$ least $X \in \mathcal{P}(\alpha) \cap L_\kappa$ such that $\forall \gamma < \beta (X \neq S_\gamma)$.

It suffices to show that $\mathcal{P}(\alpha) \cap L_\sigma \subset \{S_\beta\}_\beta = rng(S)$. Suppose there exists an $X \in \mathcal{P}(\alpha) \cap L_\sigma$ so that $\forall \beta < \kappa (S_\beta \neq X)$ and let $X_0$ denote the $<_L$-least such set. Then we see that $X_0$ is $\Sigma_1$-definable in $L_\sigma$ from $\{\alpha, \kappa\}$: there exists a $\Delta_1$-formula $\varphi(X, \alpha, \kappa)$ such that $L_\sigma \models \varphi(X_0, \alpha, \kappa) \,\&\, L_\sigma \models \exists ! X \varphi(X, \alpha, \kappa)$. By (7) we have $L_{F^{\Sigma_1}_{\alpha \cup \{\alpha,\kappa\}}(\sigma)} \models \exists X \varphi(X, \alpha, F^{\Sigma_1}_{\alpha \cup \{\alpha,\kappa\}}(\kappa))$, i.e., there exists the $<_L$-least $X_1 \in \mathcal{P}(\alpha) \cap L_{F^{\Sigma_1}_{\alpha \cup \{\alpha,\kappa\}}(\sigma)} \subset \mathcal{P}(\alpha) \cap L_\kappa$ such that $\forall \beta < F^{\Sigma_1}_{\alpha \cup \{\alpha,\kappa\}}(\kappa) (< \kappa)(S_\beta \neq X_1)$. This means that $X_1 = S_{F^{\Sigma_1}_{\alpha \cup \{\alpha,\kappa\}}(\kappa)}$. We show $X_1 = X_0$. This yields a contradiction. Denote $x \in a$ by $x \in^+ a$ and $x \notin a$ by $x \in^- a$. For any $\gamma < \alpha$, again by (7) we have

$$\begin{aligned}
&\gamma \in^{\pm} X_0 \Leftrightarrow L_\sigma \models \exists X (\gamma \in^{\pm} X \wedge \varphi(X, \alpha, \kappa)) \Rightarrow \\
&L_{F^{\Sigma_1}_{\alpha \cup \{\alpha,\kappa\}}(\sigma)} \models \exists X (\gamma \in^{\pm} X \wedge \varphi(X, \alpha, F^{\Sigma_1}_{\alpha \cup \{\alpha,\kappa\}}(\kappa))) \Leftrightarrow \gamma \in^{\pm} X_1
\end{aligned}$$

□

# 3 A theory for weakly inaccessible ordinals

Referring Theorems 2.10 and 2.12 let us interpret $\mathsf{ZF}$ to another theory. The base language here is $\{\in\}$.

In the following Definition 3.1, $I$ is intended to denote the least weakly inaccessible cardinal though we do not assume the existence of weakly inaccessible cardinals anywhere in this paper except in the **Remark** after Theorem 1.1. $\kappa, \lambda, \rho$ range over uncountable regular ordinals $< I$. The predicate $P$ is intended to denote the relation $P(\lambda, x, y)$ iff $x = F^{\Sigma_1}(\lambda; I, x \cup \{\lambda\})$ and $y = F^{\Sigma_1}(I; I, x \cup \{\lambda\})$, and the predicate $P_{I,n}(x)$ is intended to denote the relation $P_{I,n}(x)$ iff $x = F^{\Sigma_n}(I; I, x)$, where $F^{\Sigma_n}_\alpha(y) = F^{\Sigma_n}(y; I, \alpha)$ denotes the Mostowski collapsing $F^{\Sigma_n}_\alpha : \mathrm{Hull}^I_{\Sigma_n}(\alpha) \leftrightarrow L_\gamma$ of the $\Sigma_n$-Skolem hull $\mathrm{Hull}^I_{\Sigma_n}(\alpha)$ of $\alpha < I$ over $L_I$, and $F^{\Sigma_n}_\alpha(I) := \gamma$ for $L_\gamma = rng(F^{\Sigma_n}_\alpha)$.

**Definition 3.1** $\mathrm{T}(I, n)$ denotes the set theory defined as follows.

1. Its language is $\{\in, P, P_{I,n}, Reg\}$ for a ternary predicate $P$ and unary predicates $P_{I,n}$ and *Reg*.

*2.* Its axioms are obtained from those of KP$\omega$ in the expanded language [1], the axiom of constructibility $V = L$ together with the axiom schema saying that if $Reg(\kappa)$ then $\kappa$ is an uncountable regular ordinal, cf. (12) and (11), and if $P(\kappa, x, y)$ then $x$ is a critical point of the $\Sigma_1$-elementary embedding from $L_y \cong \mathrm{Hull}^I_{\Sigma_1}(x \cup \{\kappa\})$ to the universe $L_I$, cf. (11), and if $P_{I,n}(x)$ then $x$ is a critical point of the $\Sigma_n$-elementary embedding from $L_x \cong \mathrm{Hull}^I_{\Sigma_n}(x)$ to the universe $L_I$, cf.(14): for a formula $\varphi$ and an ordinal $\alpha$, $\varphi^\alpha$ denotes the result of restricting every unbounded quantifier $\exists z, \forall z$ in $\varphi$ to $\exists z \in L_\alpha, \forall z \in L_\alpha$.

*(a)* $x \in Ord$ is a $\Delta_0$-formula saying that '$x$ is an ordinal'.

$$\begin{array}{ll} & (Reg(\kappa) \to \omega < \kappa \in Ord) \\ \land & (P(\kappa, x, y) \to \{x, y\} \subset Ord \land x < y < \kappa \land Reg(\kappa)) \\ \land & (P_{I,n}(x) \to x \in Ord) \end{array} \tag{10}$$

*(b)*

$$P(\kappa, x, y) \to a \in L_x \to \varphi[\kappa, a] \to \varphi^y[x, a] \tag{11}$$

for any $\Sigma_1$-formula $\varphi$ in the language $\{\in\}$.

.

*(c)*

$$Reg(\kappa) \to a \in Ord \cap \kappa \to \exists x, y \in Ord \cap \kappa[a < x \land P(\kappa, x, y)] \tag{12}$$

*(d)*

$$\forall x \in Ord \exists y[y > x \land Reg(y)] \tag{13}$$

*(e)*

$$P_{I,n}(x) \to a \in L_x \to \varphi[a] \to \varphi^x[a] \tag{14}$$

for any $\Sigma_n$-formula $\varphi$ in the language $\{\in\}$.

*(f)*

$$a \in Ord \to \exists x \in Ord[a < x \land P_{I,n}(x)] \tag{15}$$

Let $\mathsf{ZFL}_n$ denote the subtheory of $\mathsf{ZF} + (V = L)$ obtained by restricting Separation and Collection to $\Sigma_n$-Separation and $\Sigma_n$-Collection, resp.

**Lemma 3.2** $\mathrm{T}(I) := \bigcup_{n\in\omega} \mathrm{T}(I, n)$ *is a conservative extension of Zermelo-Fraenkel set theory* $\mathsf{ZF} + (V = L)$ *with the axiom of constructibility.*

*Moreover for each* $n \geq 1$, $\mathrm{T}(I, n)$ *is a conservative extension of* $\mathsf{ZFL}_n$.

[1]This means that the predicates $P, Reg$ do not occur in $\Delta_0$-formulae for $\Delta_0$-Separation and $\Delta_0$-Collection. $P, P_{I,n}, Reg$ may occur in Foundation axiom schema.

**Proof**. Let $n \geq 1$. First consider the axioms of $\mathsf{ZFL}_n$ in $T(I,n)$. By (14), $\mathrm{T}(I,n)$ proves the reflection principle for $\Sigma_n$ $\varphi$

$$P_{I,n}(x) \to a \in L_x \to (\varphi[a] \leftrightarrow \varphi^x[a]) \tag{16}$$

Let $\varphi$ be a $\Sigma_n$-formula, and $\alpha$ an ordinal such that $\{b,c\} \subset L_\alpha$. Pick an $x$ with $\alpha < x \wedge P_{I,n}(x)$ by (15). Then by (16) $\{a \in b : \varphi[a,c]\} = \{a \in b : \varphi^x[a,c]\}$. This shows in $T(I,n)$, $\Sigma_n$-Separation from $\Delta_0$-Separation. Likewise we see that $\mathrm{T}(I,n)$ proves $\Sigma_n$-Collection.

Second consider the Power set axiom in $\mathrm{T}(I,n)$. We show that the power set $\mathcal{P}(b) = \{x : x \subset b\}$ exists as a set. Let $b \in L_\alpha$ with a multiplicative principal number $\alpha \geq \omega$. Pick a regular ordinal $\kappa > \alpha$ by (13). From Theorem 2.10 we see that ${}^\alpha\kappa \subset L_\kappa$. Let $G : Ord \to L$ be the Gödel's surjective map, which is $\Delta_1$. We have $G"\alpha = L_\alpha$ for the multiplicative principal number $\alpha$. Pick an ordinal $\beta < \alpha$ such that $G(\beta) = b$. Then ${}^\beta 2 \subset {}^\alpha\kappa \subset L_\kappa$, i.e., ${}^\beta 2 = \{x \in L_\kappa : x \in {}^\beta 2\}$, and hence by $\Delta_0$-Separation ${}^\beta 2$ exists as a set. On the other hand we have $c \in b = G(\beta) \to \exists\gamma < \beta(G(\gamma) = c)$ and $\gamma < \beta \to G(\gamma) \in G(\beta)$. Let $S : {}^\beta 2 \to \mathcal{P}(b)$ be the surjection defined by $x \in S(f)$ iff $\exists\gamma < \beta(G(\gamma) = x \wedge f(\gamma) = 1)$ for $f \in {}^\beta 2$ and $x \in b$. Pick a set $c$ such that $S"({}^\beta 2) \subset c$ by $\Delta_0$-Collection. Then $\{x : x \subset b\} = \{S(f) \in c : f \in {}^\beta 2\}$ is a set by $\Delta_0$-Separation.

Hence we have shown that $\mathsf{ZFL}_n$ is contained in $\mathrm{T}(I,n)$.

Next we show that $\mathrm{T}(I,n)$ is interpretable in $\mathsf{ZFL}_n$. Interpret the predicates $Reg(\kappa) \leftrightarrow \omega < \kappa \in Ord \wedge \forall\alpha < \kappa\forall f \in {}^\alpha\kappa[\sup_{x<\alpha} f(x) < \kappa]$ and $P(\kappa,x,y) \leftrightarrow Reg(\kappa) \wedge \{x,y\} \subset Ord \wedge (\mathrm{Hull}^I_{\Sigma_1}(x \cup \{\kappa\}) \cap \kappa \subset x) \wedge (y = \sup\{F(a) : a \in \mathrm{Hull}^I_{\Sigma_1}(x \cup \{\kappa\})\})$ for the Mostowski collapsing function $F(a) = \{F(b) : b \in \mathrm{Hull}^I_{\Sigma_1}(x \cup \{\kappa\}) \cap a\}$ and the universe $L_I = L$. Moreover for the predicate $P_{I,n}$, $P_{I,n}(x) \leftrightarrow x \in Ord \wedge (\mathrm{Hull}^I_{\Sigma_n}(x) \cap Ord \subset x)$.

We see from Theorem 2.10 that the interpreted (10), (11) and (12) are provable in $\mathsf{ZFL}_1$. Moreover the unboundedness of the regular ordinals, (13) is provable in $\mathsf{ZFL}_1$ using the Power set axiom and $\Sigma_1$-Separation.

It remains to show the interpreted (14) and (15) in $\mathsf{ZFL}_n$. It suffices to show that given an ordinal $\alpha$, there exists an ordinal $x > \alpha$ such that $\mathrm{Hull}^I_{\Sigma_n}(x) \cap Ord \subset x$. Pick a regular ordinal $\kappa > \alpha$. Again as in the proof of Proposition 1.3.2, define recursively ordinals $\{x_n\}_n$ as follows. $x_0 = \alpha + 1$, and $x_{n+1}$ is defined to be the least ordinal $x_{n+1}$ such that $\mathrm{Hull}^I_{\Sigma_n}(x_n) \cap Ord \subset x_{n+1}$. We show inductively that such an ordinal $x_n$ exists, and $x_n < \kappa$. Then $x = \sup_n x_n \leq \kappa$ is a desired one.

It suffices to show that for any $\alpha < \kappa$ there exists a $\beta < \kappa$ such that $\mathrm{Hull}^I_{\Sigma_n}(\alpha) \cap Ord \subset \beta$. By Proposition 2.3 let $h^I_{\Sigma_n}$ be be the $\Delta_n$-surjection from the $\Sigma_n$-subset $dom(h^I_{\Sigma_n})$ of $\omega \times \alpha$ to $\mathrm{Hull}^I_{\Sigma_n}(\alpha)$, which is a $\Sigma_n$-class. From $\Sigma_n$-Separation we see that $dom(h^I_{\Sigma_n})$ is a set. Hence by $\Sigma_n$-Collection, $\mathrm{Hull}^I_{\Sigma_n}(\alpha) = rng(h^I_{\Sigma_n})$ is a set. Therefore the ordinal $\sup(\mathrm{Hull}^I_{\Sigma_n}(\alpha) \cap Ord)$ exists in the universe. On the other hand we have for the subset $dom(h^I_{\Sigma_n})$ of $\omega\times\alpha$, $dom(h^I_{\Sigma_n}) \in L_\kappa$ by Theorem 2.12. Hence $\kappa \leq \sup(\mathrm{Hull}^I_{\Sigma_n}(\alpha) \cap Ord)$ would yield a cofinal map from $\alpha$ to $\kappa$, which is a subset of the set $h^I_{\Sigma_n}$ in the universe.

This contradicts the regularity of $\kappa$. Therefore $\sup(\mathrm{Hull}^{I}_{\Sigma_n}(\alpha) \cap Ord) < \kappa$. $\Box$

## 4 Ordinals for inaccessibles

Let $Ord^{\varepsilon}$ and $<^{\varepsilon}$ be $\Delta$-predicates such that for any wellfounded model $M$ of KP$\omega$, $<^{\varepsilon}$ is a well ordering of type $\varepsilon_{I+1}$ on $Ord^{\varepsilon}$ for the order type $I$ of the class $Ord$ in $M$. $\lceil \omega_n(I+1) \rceil \in Ord^{\varepsilon}$ denotes the code of the 'ordinal' $\omega_n(I+1)$, which is assumed to be a closed 'term' built from the code $\lceil I \rceil$ and $n$, e.g., $\lceil \alpha \rceil = \langle 0, \alpha \rangle$ for $\alpha \in Ord$, $\lceil I \rceil = \langle 1, 0 \rangle$ and $\lceil \omega_n(I+1) \rceil = \langle 2, \langle 2, \cdots \langle 2, \langle 3, \lceil I \rceil, \langle 0, 1 \rangle \rangle \rangle \cdots \rangle \rangle$. For simplicity let us identify the code $\lceil \alpha \rceil \in Ord^{\varepsilon}$ with the 'ordinal' $\alpha < \varepsilon_{I+1}$, and $<^{\varepsilon}$ is denoted by $<$ when no confusion likely occurs.

$<$, i.e., $<^{\varepsilon}$ is assumed to be a canonical ordering such that $\mathsf{KP}\omega$ proves the fact that $<$ is a linear ordering, and for any formula $\varphi$ and each $n < \omega$,

$$\mathsf{KP}\omega \vdash \forall x(\forall y < x\, \varphi(y) \to \varphi(x)) \to \forall x < \omega_n(I+1)\varphi(x) \tag{17}$$

In what follows of this section $n \geq 1$ denotes a fixed positive integer, and we work in $\mathsf{ZF} + (V = L)$.

As before, $I$ (or its code $\lceil I \rceil = \langle 1, 0 \rangle$) is intended to denote the least weakly inaccessible ordinal. $R$ denotes the set of uncountable regular ordinals$< I$, while $R^{+} := R \cup \{I\}$. $\kappa, \lambda, \rho$ denote elements of $R$.

Define simultaneously by recursion on ordinals $\alpha < \omega_{n+1}(I+1)$, the classes $\mathcal{H}_{\alpha,n}(X) \subset L_{\omega_{n+1}(I+1)}\, (X \subset L_{\omega_{n+1}(I+1)})$ and the ordinals $\Psi_{\kappa,n}\alpha\, (\kappa \in R)$ and $\Psi_{I,n}\alpha$ as follows.

**Definition 4.1** $\mathcal{H}_{\alpha,n}(X)$ is the Skolem hull of $\{0, I\} \cup X$ under the functions $+, \alpha \mapsto \omega^{\alpha} < \omega_{n+1}(I+1), \Psi_{I,n}\restriction\alpha, \Psi_{\kappa,n}\restriction\alpha\, (\kappa \in R)$, the $\Sigma_n$-definability, and the Mostowski collapsing functions $(x, \kappa, d) \mapsto F^{\Sigma_1}_{x \cup \{\kappa\}}(d)\, (\kappa \in R, \mathrm{Hull}^{I}_{\Sigma_1}(x \cup \{\kappa\}) \cap \kappa \subset x)$ and $(x, d) \mapsto F^{\Sigma_n}_{x}(d)\, (\mathrm{Hull}^{I}_{\Sigma_n}(x) \cap I \subset x)$.

For a later reference let us define stages $\mathcal{H}^{m}_{\alpha,n}(X)\, (m \in \omega)$ of the inductive definition.

1. $\mathcal{H}^{0}_{\alpha,n}(X) = \{0, I\} \cup X$.
2. $x, y \in \mathcal{H}^{m}_{\alpha,n}(X) \cap \omega_{n+1}(I+1) \Rightarrow x + y \in \mathcal{H}^{m+1}_{\alpha,n}(X)$.
   $x \in \mathcal{H}^{m}_{\alpha,n}(X) \cap \omega_{n}(I+1) \Rightarrow \omega^{x} \in \mathcal{H}^{m+1}_{\alpha,n}(X)$.
3. $\gamma \in \mathcal{H}^{m}_{\alpha,n}(X) \cap \alpha \Rightarrow \Psi_{I,n}\gamma \in \mathcal{H}^{m+1}_{\alpha,n}(X)$.
4. $\kappa \in \mathcal{H}^{m}_{\alpha,n}(X) \cap R \,\&\, \gamma \in \mathcal{H}^{m}_{\alpha,n}(X) \cap \alpha \Rightarrow \Psi_{\kappa,n}\gamma \in \mathcal{H}^{m+1}_{\alpha,n}(X)$.
5. $\mathrm{Hull}^{I}_{\Sigma_n}(\mathcal{H}_{\alpha,n}(X) \cap L_I) \subset \mathcal{H}_{\alpha,n}(X)$.

   Namely for any $\Sigma_n$-formula $\varphi[x, \vec{y}]$ in the language $\{\in\}$ and parameters $\vec{a} \subset \mathcal{H}^{m}_{\alpha,n}(X) \cap L_I$, if $b \in L_I$, $L_I \models \varphi[b, \vec{a}]$ and $L_I \models \exists! x\, \varphi[x, \vec{a}]$, then $b \in \mathcal{H}^{m+1}_{\alpha,n}(X)$.

6. If $\kappa \in \mathcal{H}^m_{\alpha,n}(X) \cap R$, $x \in \mathcal{H}^m_{\alpha,n}(X) \cap \kappa$ with $\mathrm{Hull}^I_{\Sigma_1}(x \cup \{\kappa\}) \cap \kappa \subset x$ and $(\kappa = \omega_{c+1} \Rightarrow \omega_c < x)$, and $d \in (\mathrm{Hull}^I_{\Sigma_1}(x \cup \{\kappa\}) \cup \{I\}) \cap \mathcal{H}^m_{\alpha,n}(X)$, then $F^{\Sigma_1}_{x\cup\{\kappa\}}(d) \in \mathcal{H}^{m+1}_{\alpha,n}(X)$.

7. If $x \in \mathcal{H}^m_{\alpha,n}(X) \cap I$ with $\mathrm{Hull}^I_{\Sigma_n}(x) \cap I \subset x$, and $d \in (\mathrm{Hull}^I_{\Sigma_n}(x) \cup \{I\}) \cap \mathcal{H}^m_{\alpha,n}(X)$, then $F^{\Sigma_n}_x(d) \in \mathcal{H}^{m+1}_{\alpha,n}(X)$.

8. $\mathcal{H}_{\alpha,n}(X) := \bigcup\{\mathcal{H}^m_{\alpha,n}(X) : m \in \omega\}$.

For $\kappa \in R^+$

$$\Psi_{\kappa,n}\alpha := \min\{\beta \le \kappa : \kappa \in \mathcal{H}_{\alpha,n}(\beta) \,\&\, \mathcal{H}_{\alpha,n}(\beta) \cap \kappa \subset \beta\}.$$

The ordinal $\Psi_{\kappa,n}\alpha$ is well defined and $\Psi_{\kappa,n}\alpha \le \kappa$ for any uncountable regular $\kappa \le I$ since $\kappa \in \mathcal{H}_{\alpha,n}(\kappa)$ by Proposition 4.4.1 below.

**Proposition 4.2** *1. $\mathcal{H}_{\alpha,n}(X)$ is closed under $\Sigma_n$-definability: $\vec{a} \subset \mathcal{H}_{\alpha,n}(X) \cap L_I \Rightarrow \mathrm{Hull}^I_{\Sigma_n}(\vec{a}) \cap L_I \subset \mathcal{H}_{\alpha,n}(X)$.*

*2. For $\kappa \in R$, $\mathrm{Hull}^I_{\Sigma_1}(\Psi_{\kappa,n}\alpha \cup \{\kappa\}) \cap \kappa = \Psi_{\kappa,n}\alpha$. Namely $\Psi_{\kappa,n}\alpha \in Cr^I_{\Sigma_1}(\{\kappa\})$.*

*3. $\mathcal{H}_{\alpha,n}(X)$ is closed under the Veblen function $\varphi$ on $I$, $x, y \in \mathcal{H}_{\alpha,n}(X) \cap I \Rightarrow \varphi xy \in \mathcal{H}_{\alpha,n}(X)$.*

*4. If $\kappa \in \mathcal{H}_{\alpha,n}(X) \cap R$, $x \in \mathcal{H}_{\alpha,n}(X) \cap \kappa$, $\mathrm{Hull}^I_{\Sigma_1}(x \cup \{\kappa\}) \cap \kappa \subset x$, $(\kappa = \omega_{c+1} \Rightarrow \omega_c < x)$ and $\delta \in (\mathrm{Hull}^I_{\Sigma_1}(x \cup \{\kappa\}) \cup \{I\}) \cap \mathcal{H}_{\alpha,n}(X)$, then $F^{\Sigma_1}_{x\cup\{\kappa\}}(\delta) \in \mathcal{H}_{\alpha,n}(X)$.*

*5. If $x \in \mathcal{H}_{\alpha,n}(X) \cap I$, $\mathrm{Hull}^I_{\Sigma_n}(x) \cap I \subset x$ and $\delta \in (\mathrm{Hull}^I_{\Sigma_n}(x) \cup \{I\}) \cap \mathcal{H}_{\alpha,n}(X)$, then $F^{\Sigma_n}_x(\delta) \in \mathcal{H}_{\alpha,n}(X)$.*

*6. Assume $n \ge 2$. $\gamma \in \mathcal{H}_{\alpha,n}(X) \cap I \Leftrightarrow \omega_\gamma \in \mathcal{H}_{\alpha,n}(X) \cap I$ for $\omega_\alpha = \aleph_\alpha$.*
*Moreover $\gamma \in \mathcal{H}_{\alpha,n}(X) \cap I \Rightarrow \gamma^+ = \min\{\lambda \in R : \gamma < \lambda\} \in \mathcal{H}_{\alpha,n}(X) \cap I$ .*

**Proof**. 4.2.2. By the definition of $\mathcal{H}_{\alpha,n}(X)$, we have

$$\mathrm{Hull}^I_{\Sigma_1}(\Psi_{\kappa,n}\alpha \cup \{\kappa\}) \cap \kappa \subset \mathcal{H}_{\alpha,n}(\Psi_{\kappa,n}\alpha) \cap \kappa \subset \Psi_{\kappa,n}\alpha \subset \mathrm{Hull}^I_{\Sigma_1}(\Psi_{\kappa,n}\alpha \cup \{\kappa\}) \cap \kappa.$$

4.2.3. This is seen from the $\Sigma_1$-definability of the Veblen function $\varphi$.
4.2.6. From Corollary 2.11.2 the map $I > \alpha \mapsto \omega_\alpha$ and its inverse are $\Delta_2$-definable. Moreover the next regular ordinal $\gamma^+$ is $\Delta_2$-definable. □

In the following Proposition 4.3, for $\kappa \in R^+$ and $x$, $(\mathrm{Hull}(x,\kappa), F_{x,\kappa})$ denotes $(\mathrm{Hull}^I_{\Sigma_1}(x \cup \{\kappa\}), F^{\Sigma_1}_{x\cup\{\kappa\}})$ when $\kappa < I$, and $(\mathrm{Hull}^I_{\Sigma_n}(x), F^{\Sigma_n}_x)$ when $\kappa = I$.

**Proposition 4.3** *Suppose $n \ge 2$, $\kappa, \lambda \in R^+$, $\mathrm{Hull}(x,\kappa) \cap \kappa \subset x$, and $\omega_c < x$ if $\kappa = \omega_{c+1}$. Then $x < \Psi_{\lambda,n}b \Rightarrow F_{x,\kappa}(I) < \Psi_{\lambda,n}b$, and $a \in \mathcal{H}_{b,n}(\Psi_{\kappa,n}b) \cap b \Rightarrow \Psi_{\kappa,n}a < \Psi_{\kappa,n}b$.*

**Proof**. Suppose $x < \Psi_{\lambda,n} b$. We show $\kappa \in \mathcal{H}_{b,n}(\Psi_{\lambda,n} b)$. If $\kappa = I$, there is nothing to show. If $\kappa = \omega_{c+1}$, we have $c \leq \omega_c < x < \Psi_{\lambda,n} b$ and $c \in \mathcal{H}_{b,n}(\Psi_{\lambda,n} b)$. By Proposition 4.2.6 we have $\kappa = \omega_{c+1} \in \mathcal{H}_{b,n}(\Psi_{\lambda,n} b)$. Thus $F_{x,\kappa}(I) \in \mathcal{H}_{b,n}(\Psi_{\lambda,n} b)$. It remains to see $y := F_{x,\kappa}(I) < \lambda$. We have a definable bijection from $x$ to $L_y$. Since $x < \lambda$, we conclude $F_{x,\kappa}(I) = y < \lambda$. □

We see the following Proposition 4.4 as in [6].

**Proposition 4.4** *Let* $n \geq 2$.

1. *For any* $\kappa \in R^+$, $\kappa \in \mathcal{H}_{\alpha,n}(\kappa)$, $\kappa \in \mathcal{H}_{\alpha,n}(\Psi_{\kappa,n}\alpha)$ *and* $\Psi_{\kappa,n}\alpha < \kappa$.
2. $\Psi_{\kappa,n}\alpha \notin \{\omega_\beta : \beta < \omega_\beta\}$.
3. $\omega_a < \Psi_{\omega_{a+1},n}\alpha < \omega_{a+1}$.
4. $\omega_{\Psi_{I,n}\alpha} = \Psi_{I,n}\alpha$.
5. $\Psi_{I,n}\alpha < I$

The following Proposition 4.5 is easy to see.

**Proposition 4.5** *Both of* $x = \mathcal{H}_{\alpha,n}(\beta)\,(\alpha < \omega_{n+1}(I+1), \beta < I)$ *and* $y = \Psi_{\kappa,n}\alpha\,(\kappa \in R^+)$ *are* $\Sigma_{n+1}$*-predicates as fixed points in* $\mathsf{ZF}$.

**Lemma 4.6** $\forall \alpha < \omega_{n+1}(I+1) \forall \kappa \in R^+ \exists x < \kappa[x = \Psi_{\kappa,n}\alpha]$.

**Proof**. By Proposition 4.5 both $x = \mathcal{H}_{\alpha,n}(\beta)\,(\alpha < \omega_{n+1}(I+1), \beta < I)$ and $y = \Psi_{\kappa,n}\alpha\,(\kappa \in R^+)$ are $\Sigma_{n+1}$-predicates. We show that $A(\alpha) :\Leftrightarrow \forall \beta < I \exists x[x = \mathcal{H}_{\alpha,n}(\beta)] \land \forall \kappa \in R^+ \exists \beta < \kappa[\Psi_{\kappa,n}\alpha = \beta]$ is progressive along $<$. Then $\forall \alpha < \omega_{n+1}(I+1) \forall \kappa \in R^+ \exists x < \kappa[x = \Psi_{\kappa,n}\alpha]$ will follow from transfinite induction up to $\omega_{n+1}(I+1)$, cf. (17).

Assume $\forall \gamma < \alpha\, A(\gamma)$ as our IH. We have $\forall x \exists h[h = \mathrm{Hull}^I_{\Sigma_n}(x)]$. We see from this, IH and Separation that $\forall X \exists ! Y\, D_{\alpha,n}(X,Y)$, where $D_{\alpha,n}(X,Y)$ is a $\Sigma_{n+1}$-predicate such that if $D_{\alpha,n}(\mathcal{H}^m_{\alpha,n}(\beta), Y)$ then $Y = \mathcal{H}^{m+1}_{\alpha,n}(\beta)$ for any $Y$. Therefore $\forall \beta < I \exists x[x = \mathcal{H}_{\alpha,n}(\beta) = \bigcup_m \mathcal{H}^m_{\alpha,n}(\beta)]$.

Next as in the Proof of Theorem 2.10, (3) $\Rightarrow$ (4), define recursively ordinals $\{\beta_m\}_m$ for $\kappa \in R^+$ as follows. $\beta_0 = 0$ if $\kappa = I$ and $\beta_0 = a+1$ if $\kappa = \omega_{a+1}$, and $\beta_{m+1}$ is defined to be the least ordinal $\beta_{m+1} \leq \kappa$ such that $\mathcal{H}_{\alpha,n}(\beta_m) \cap \kappa \subset \beta_{m+1}$.

We see inductively that $\beta_m < \kappa$ using the regularity of $\kappa$ and the facts that $\forall \beta < I \exists x[x = \mathcal{H}_\alpha(\beta)]$ and $\forall \beta < \kappa \exists x[x = \mathcal{H}_\alpha(\beta) \land card(x) < \kappa]$ for $\kappa = \omega_{a+1}$. For the case $\kappa = I$, $card(x) < I$ can be replaced by $card(x) < \omega_1$. The latter follows from the fact that $\forall X \exists Y[D_{\alpha,n}(X,Y) \land \{card(X) < \kappa \to card(Y) < \kappa\}]$.

Moreover $m \mapsto \beta_m$ is a definable map. Therefore $\beta = \sup_m \beta_m < \kappa$ enjoys $\mathcal{H}_{\alpha,n}(\beta) \cap \kappa \subset \beta$. Also $a \in \mathcal{H}_{\alpha,n}(\beta)$ for $\kappa = \omega_{a+1}$. □

# 5 Operator controlled derivations for weakly inaccessibles

This section relies on Buchholz' techniques in [6].

In what follows of this section $n \geq 2$ denotes a fixed positive integer, and we work in $\mathsf{ZF} + (V = L)$. We consider only the ordinals $< \omega_{n+1}(I+1)$.

$L = L_I = \bigcup_{\alpha<I} L_\alpha$ denotes the universe. Both $L_I \models A$ and '$A$ is true' are synonymous with $A$.

## 5.1 Classes of formulae

The language $\mathcal{L}_c$ is obtained from $\{\in, P, P_{I,n}, Reg\}$ by adding names(individual constants) $c_a$ of each set $a \in L$. $c_a$ is identified with $a$. A *term* in $\mathcal{L}_c$ is either a variable or a constant in $L$.

Formulae in this language are defined in the next definition. Formulae are assumed to be in negation normal form.

**Definition 5.1** *1.* Let $t_1, \ldots, t_m$ be terms. For each $m$-ary predicate constant $R \in \{\in, P, P_{I,n}, Reg\}$ $R(t_1, \ldots, t_m)$ and $\neg R(t_1, \ldots, t_m)$ are formulae, where $m = 1, 2, 3$. These are called *literals.*

*2.* If $A$ and $B$ are formulae, then so are $A \wedge B$ and $A \vee B$.

*3.* Let $t$ be a term. If $A$ is a formula and the variable $x$ does not occur in $t$, then $\exists x \in t\, A$ and $\forall x \in t\, A$ are *bounded* formulae.

*4.* If $A$ is a formula and $x$ a variable, then $\exists x\, A$ and $\forall x\, A$ are *unbounded* formulae. Unbounded quantifiers $\exists x, \forall x$ are denoted by $\exists x \in L_I, \forall x \in L_I$, resp.

For formulae $A$ in $\mathcal{L}_c$, $\mathsf{qk}(A)$ denotes the finite set of sets $a$ which are bounds of 'bounded' quantifiers $\exists x \in a, \forall x \in a$ occurring in $A$. Moreover $\mathsf{k}(A)$ denotes the set of sets occurring in $A$. $\mathsf{k}(A)$ is defined to include bounds of 'bounded' quantifiers. By definition we set $0 \in \mathsf{qk}(A)$. Thus $0 \in \mathsf{qk}(A) \subset \mathsf{k}(A) \subset L_I \cup \{L_I\}$.

**Definition 5.2** *1.* $\mathsf{k}(\neg A) = \mathsf{k}(A)$ and similarly for $\mathsf{qk}$.

*2.* $\mathsf{qk}(M) = \{0\}$ for any literal $M$.

*3.* $\mathsf{k}(Q(t_1, \ldots, t_m)) = (\{t_1, \ldots, t_m\} \cap L_I) \cup \{0\}$ for literals $Q(t_1, \ldots, t_m)$ with predicates $Q$ in the set $\{\in, P, P_{I,n}, Reg\}$.

*4.* $\mathsf{k}(A_0 \vee A_1) = \mathsf{k}(A_0) \cup \mathsf{k}(A_1)$ and similarly for $\mathsf{qk}$.

*5.* For $a \in L_I \cup \{L_I\}$, $\mathsf{k}(\exists x \in a\, A(x)) = \{a\} \cup \mathsf{k}(A(x))$ and similarly for $\mathsf{qk}$.

*6.* For variables $y$, $\mathsf{k}(\exists x \in y\, A(x)) = \mathsf{k}(A(x))$ and similarly for $\mathsf{qk}$.

*7.* For sets $\Gamma$ of formulae $\mathsf{k}(\Gamma) := \bigcup\{\mathsf{k}(A) : A \in \Gamma\}$.

For example $\mathsf{qk}(\exists x \in a\, A(x)) = \{a\} \cup \mathsf{qk}(A(x))$.

**Definition 5.3** For $a \in L_I \cup \{L_I\}$, $\mathrm{rk}_L(a)$ denotes the *$L$-rank* of $a$.

$$\mathrm{rk}_L(a) := \begin{cases} \min\{\alpha \in Ord : a \in L_{\alpha+1}\} & a \in L_I = L \\ I & a = L_I \end{cases}$$

**Definition 5.4** *1.* $A \in \Delta_0$ iff there exists a $\Delta_0$-formula $\theta[\vec{x}]$ in the language $\{\in\}$ and terms $\vec{t}$ in $\mathcal{L}_c$ such that $A \equiv \theta[\vec{t}]$. This means that $A$ is bounded, and the predicates $P, P_{I,n}, Reg$ do not occur in $A$.

*2.* Putting $\Sigma_0 := \Pi_0 := \Delta_0$, the classes $\Sigma_m$ and $\Pi_m$ of formulae in the language $\mathcal{L}_c$ are defined as usual, where by definition $\Sigma_m \cup \Pi_m \subset \Sigma_{m+1} \cap \Pi_{m+1}$.

Each formula in $\Sigma_m \cup \Pi_m$ is in prenex normal form with alternating unbounded quantifiers and $\Delta_0$-matrix.

*3.* The set $\Sigma^{\Sigma_{n+1}}(\lambda)$ of sentences is defined recursively as follows. Let $\{a, b, c\} \subset L_I$ and $d \in L_I \cup \{L_I\}$.

*(a)* Each $\Sigma_{n+1}$-sentence is in $\Sigma^{\Sigma_{n+1}}(\lambda)$.

*(b)* Each literal including $Reg(a), P(a, b, c), P_{I,n}(a)$ and its negation is in $\Sigma^{\Sigma_{n+1}}(\lambda)$.

*(c)* $\Sigma^{\Sigma_{n+1}}(\lambda)$ is closed under propositional connectives $\vee, \wedge$.

*(d)* Suppose $\forall x \in d\, A(x) \not\in \Delta_0$. Then $\forall x \in d\, A(x) \in \Sigma^{\Sigma_{n+1}}(\lambda)$ iff $A(\emptyset) \in \Sigma^{\Sigma_{n+1}}(\lambda)$ and $\mathrm{rk}_L(d) < \lambda$.

*(e)* Suppose $\exists x \in d\, A(x) \not\in \Delta_0$. Then $\exists x \in d\, A(x) \in \Sigma^{\Sigma_{n+1}}(\lambda)$ iff $A(\emptyset) \in \Sigma^{\Sigma_{n+1}}(\lambda)$ and $\mathrm{rk}_L(d) \leq \lambda$.

Note that the predicates $P, P_{I,n}, Reg$ do not occur in $\Sigma_m$-formulae.

**Definition 5.5** Let us extend the domain $dom(F^{\Sigma_1}_{x\cup\{\kappa\}}) = \mathrm{Hull}^I_{\Sigma_1}(x \cup \{\kappa\})$ of the Mostowski collapse to formulae.

$$dom(F^{\Sigma_1}_{x\cup\{\kappa\}}) = \{A \in \Sigma_1 \cup \Pi_1 : \mathsf{k}(A) \subset \mathrm{Hull}^I_{\Sigma_1}(x \cup \{\kappa\}) \cup \{I\}\}.$$

For $A \in dom(F^{\Sigma_1}_{x\cup\{\kappa\}})$, $F^{\Sigma_1}_{x\cup\{\kappa\}}$"$A$ denotes the result of replacing each constant $c \in L_I$ by $F^{\Sigma_1}_{x\cup\{\kappa\}}(c)$, each unbounded existential quantifier $\exists z \in L_I$ by $\exists z \in L_{F^{\Sigma_1}_{x\cup\{\kappa\}}(I)}$, and each unbounded universal quantifier $\forall z \in L_I$ by $\forall z \in L_{F^{\Sigma_1}_{x\cup\{\kappa\}}(I)}$.

For sequent, i.e., finite set of sentences $\Gamma \subset dom(F^{\Sigma_1}_{x\cup\{\kappa\}})$, put $F^{\Sigma_1}_{x\cup\{\kappa\}}$"$\Gamma = \{F^{\Sigma_1}_{x\cup\{\kappa\}}$"$A : A \in \Gamma\}$.

Likewise the domain $dom(F^{\Sigma_n}_x) = \mathrm{Hull}^I_{\Sigma_n}(x)$ is extended to

$$dom(F^{\Sigma_n}_x) = \{A \in \Sigma_n \cup \Pi_n : \mathsf{k}(A) \subset \mathrm{Hull}^I_{\Sigma_n}(x) \cup \{I\}\}$$

and for formula $A \in dom(F^{\Sigma_n}_x)$, $F^{\Sigma_n}_x$"$A$, and sequent $\Gamma \subset dom(F^{\Sigma_n}_x)$, $F^{\Sigma_n}_x$"$\Gamma$ are defined similarly.

**Proposition 5.6** *For $F = F^{\Sigma_1}_{x\cup\{\kappa\}}, F^{\Sigma_n}_x$ and $A \in dom(F)$, $A \leftrightarrow F"A$.*

The assignment of disjunctions and conjunctions to sentences is defined as in [6] *except* for $\Sigma_n \cup \Pi_n$-formulae.

**Definition 5.7** *1.* If $M$ is one of the literals $a \in b, a \notin b$, then for $J := 0$

$$M :\simeq \begin{cases} \bigvee(A_\iota)_{\iota\in J} & \text{if } M \text{ is false (in } L) \\ \bigwedge(A_\iota)_{\iota\in J} & \text{if } M \text{ is true} \end{cases}$$

*2.* $(A_0 \vee A_1) :\simeq \bigvee(A_\iota)_{\iota\in J}$ and $(A_0 \wedge A_1) :\simeq \bigwedge(A_\iota)_{\iota\in J}$ for $J := 2$.

*3.* $Reg(a) :\simeq \bigvee(a \notin a)_{\iota\in J}$ and $\neg Reg(a) :\simeq \bigwedge(a \in a)_{\iota\in J}$ with $J := \begin{cases} 1 & \text{if } a \in R \\ 0 & \text{otherwise} \end{cases}$.

*4.* $P(a,b,c) :\simeq \bigvee(a \notin a)_{\iota\in J}$ and $\neg P(a,b,c) :\simeq \bigwedge(a \in a)_{\iota\in J}$ with

$$J := \begin{cases} 1 & \text{if } a \in R \,\&\, \exists\alpha < \omega_{n+1}(I+1)[b = \Psi_{a,n}\alpha \,\&\, \alpha \in \mathcal{H}_{\alpha,n}(b) \,\&\, c = F^{\Sigma_1}_{b\cup\{a\}}(I)] \\ 0 & \text{otherwise} \end{cases}.$$

*5.* $P_{I,n}(a) :\simeq \bigvee(a \notin a)_{\iota\in J}$ and $\neg P_{I,n}(a) :\simeq \bigwedge(a \in a)_{\iota\in J}$ with

$$J := \begin{cases} 1 & \text{if } \exists\alpha < \omega_{n+1}(I+1)[a = \Psi_{I,n}\alpha \,\&\, \alpha \in \mathcal{H}_{\alpha,n}(a)] \\ 0 & \text{otherwise} \end{cases}.$$

*6.* Let $\exists z \in b\, \theta[z] \in \Sigma_n$ for $b \in L_I \cup \{L_I\}$. Then for the set

$$d := \mu z \in b\, \theta[z] := \min_{<_L}\{d : (d \in b \wedge \theta[d]) \vee (\neg\exists z \in b\, \theta[z] \wedge d = 0)\} \tag{18}$$

with a canonical well ordering $<_L$ on $L$, and $J = \{d\}$

$$\begin{aligned} \exists z \in b\, \theta[z] \quad &:\simeq \quad \bigvee(d \in b \wedge \theta[d])_{d\in J} \\ \forall z \in b\, \neg\theta[z] \quad &:\simeq \quad \bigwedge(d \in b \to \neg\theta[d])_{d\in J} \end{aligned} \tag{19}$$

where $d \in b$ denotes a true literal, e.g., $d \notin d$ when $b = L_I$.

*7.* Otherwise set for $a \in L_I \cup \{L_I\}$ and $J := \{b : b \in a\}$

$$\exists x \in a\, A(x) :\simeq \bigvee(A(b))_{b\in J} \text{ and } \forall x \in a\, A(x) :\simeq \bigwedge(A(b))_{b\in J}.$$

The rank $\text{rk}(A)$ of sentences $A$ is defined by recursion on the number of symbols occurring in $A$.

**Definition 5.8** *1.* $\text{rk}(\neg A) := \text{rk}(A)$.

*2.* $\text{rk}(a \in b) := 0$.

*3.* $\text{rk}(Reg(a)) := \text{rk}(P(a,b,c)) := \text{rk}(P_{I,n}(a)) := 1$.

*4.* $\mathrm{rk}(A_0 \lor A_1) := \max\{\mathrm{rk}(A_0), \mathrm{rk}(A_1)\} + 1$.

*5.* $\mathrm{rk}(\exists x \in a\, A(x)) := \max\{\omega\alpha, \mathrm{rk}(A(\emptyset)) + 2\}$ for $\alpha = \mathrm{rk}_L(a)$.

**Proposition 5.9** *Let* $A \simeq \bigvee(A_\iota)_{\iota\in J}$ *or* $A \simeq \bigwedge(A_\iota)_{\iota\in J}$.

1. $\forall\iota \in J(\mathsf{k}(A_\iota) \subset \mathsf{k}(A) \cup \{\iota\})$.
2. $A \in \Sigma^{\Sigma_{n+1}}(\lambda) \Rightarrow \forall\iota \in J(A_\iota \in \Sigma^{\Sigma_{n+1}}(\lambda))$.
3. *For an ordinal* $\lambda \leq I$ *with* $\omega\lambda = \lambda$, $\mathrm{rk}(A) < \lambda \Rightarrow A \in \Sigma^{\Sigma_{n+1}}(\lambda)$.
4. $\mathrm{rk}(A) < I + \omega$.
5. $\mathrm{rk}(A) \in \{\omega\, \mathrm{rk}_L(a) + i : a \in \mathsf{qk}(A), i \in \omega\} \subset \mathrm{Hull}^I_{\Sigma_1}(\mathsf{k}(A))$.
6. $\forall\iota \in J(\mathrm{rk}(A_\iota) < \mathrm{rk}(A))$.

**Proof**. 5.9.6. This is seen from the fact that $a \in b \in L_I \cup \{L_I\} \Rightarrow \mathrm{rk}_L(a) < \mathrm{rk}_L(b)$. $\Box$

## 5.2 Operator controlled derivations

Let $\mathcal{H}$ be an operator $\mathcal{H} : \mathcal{P}(L_{\omega_{n+1}(I+1)}) \to \mathcal{P}(L_{\omega_{n+1}(I+1)})$ on $L_{\omega_{n+1}(I+1)}$. For $\Theta \in \mathcal{P}(L_{\omega_{n+1}(I+1)})$, $\mathcal{H}[\Theta]$ denotes the operator defined by $\mathcal{H}[\Theta](X) := \mathcal{H}(\Theta\cup X)$ for $X \in \mathcal{P}(L_{\omega_{n+1}(I+1)})$. The map $X \mapsto \mathcal{H}_{\alpha,n}(X)$ defined in Definition 4.1 is an example of an operator on $L_{\omega_{n+1}(I+1)}$.

Let $\mathcal{H}$ be an operator $\mathcal{H}$ on $L_{\omega_{n+1}(I+1)}$, $\kappa \in R^+$, $\Gamma$ a sequent, $a < \omega_{n+1}(I+1)$ and $b < I + \omega$. By recursion on ordinals $a$ we define a relation $(\mathcal{H}, \kappa, n) \vdash^a_b \Gamma$, which is read 'there exists an infinitary derivation of $\Gamma$ which is $(\kappa, n)$-*controlled* by $\mathcal{H}$, and whose height is at most $a$ and its cut rank is less than $b$'.

Sequents are finite sets of sentences, and inference rules are formulated in one-sided sequent calculus.

**Definition 5.10** By recursion on ordinals $a$ define a relation $(\mathcal{H}, \kappa, n) \vdash^a_b \Gamma$ as follows.

$(\mathcal{H}, \kappa, n) \vdash^a_b \Gamma$ holds if

$$\{a\} \cup \mathsf{k}(\Gamma) \subset \mathcal{H} := \mathcal{H}(\emptyset) \tag{20}$$

and one of the following cases holds:

($\bigvee$) $A \simeq \bigvee\{A_\iota : \iota \in J\}$, $A \in \Gamma$ and there exist $\iota \in J$ and $a(\iota) < a$ such that

$$\mathrm{rk}_L(\iota) < \kappa \Rightarrow \mathrm{rk}_L(\iota) < a \tag{21}$$

and $(\mathcal{H}, \kappa, n) \vdash^{a(\iota)}_b \Gamma, A_\iota$.

($\bigwedge$) $A \simeq \bigwedge\{A_\iota : \iota \in J\}$, $A \in \Gamma$ and for every $\iota \in J$ there exists an $a(\iota) < a$ such that $(\mathcal{H}[\{\iota\}], \kappa, n) \vdash^{a(\iota)}_b \Gamma, A_\iota$.

($cut$) There exist $a_0 < a$ and $C$ such that $\mathrm{rk}(C) < b$ and $(\mathcal{H},\kappa,n) \vdash^{a_0}_{b} \Gamma, \neg C$ and $(\mathcal{H},\kappa,n) \vdash^{a_0}_{b} C, \Gamma$.

$(\mathbf{P}_\lambda)$ $\lambda \in R$ and there exists $\alpha < \lambda$ such that $(\exists x, y < \lambda[\alpha < x \wedge P(\lambda, x, y)]) \in \Gamma$.

$(\mathbf{F}^{\Sigma_1}_{x\cup\{\lambda\}})$ $\lambda \in \mathcal{H} \cap R$, $x = \Psi_{\lambda,n}\beta \in \mathcal{H}$ for a $\beta$ and there exist $a_0 < a$, $\Gamma_0 \subset \Sigma_1$ and $\Lambda$ such that $\mathsf{k}(\Gamma_0) \subset \mathrm{Hull}^{I}_{\Sigma_1}((\mathcal{H} \cap x) \cup \{\lambda\}) \cup \{I\}$, $\Gamma = \Lambda \cup (F^{\Sigma_1}_{x\cup\{\lambda\}}{}''\Gamma_0)$ and $(\mathcal{H},\kappa,n) \vdash^{a_0}_{b} \Lambda, \Gamma_0$, where $F^{\Sigma_1}_{x\cup\{\lambda\}}$ denotes the Mostowski collapse $F^{\Sigma_1}_{x\cup\{\lambda\}} : \mathrm{Hull}^{I}_{\Sigma_1}(x \cup \{\lambda\}) \leftrightarrow L_{F^{\Sigma_1}_{x\cup\{\lambda\}}(I)}$.

$(\mathbf{P}_{I,n})$ There exists $\alpha < I$ such that $(\exists x < I[\alpha < x \wedge P_{I,n}(x)]) \in \Gamma$.

$(\mathbf{F}^{\Sigma_n}_{x})$ $x = \Psi_{I,n}\beta \in \mathcal{H}$ for a $\beta$ and there exist $a_0 < a$, $\Gamma_0 \subset \Sigma_n$ and $\Lambda$ such that $\mathsf{k}(\Gamma_0) \subset \mathrm{Hull}^{I}_{\Sigma_n}(\mathcal{H} \cap x) \cup \{I\}$, $\Gamma = \Lambda \cup (F^{\Sigma_n}_{x}{}''\Gamma_0)$ and $(\mathcal{H},\kappa,n) \vdash^{a_0}_{b} \Lambda, \Gamma_0$, where $F^{\Sigma_n}_{x}$ denotes the Mostowski collapse $F^{\Sigma_n}_{x} : \mathrm{Hull}^{I}_{\Sigma_n}(x) \leftrightarrow L_{F^{\Sigma_n}_{x}(I)}$.

**Proposition 5.11** $(\mathcal{H},\kappa,n) \vdash^{a}_{b} \Gamma \,\&\, \lambda \leq \kappa \Rightarrow (\mathcal{H},\lambda,n) \vdash^{a}_{b} \Gamma$.

The inferences rules $(\bigvee)$, $(\bigwedge)$ and ($cut$) are standard except $\Sigma_n \cup \Pi_n$-formulae are derived from specific minor formulae, (19). $(\mathbf{P}_\lambda)$ is an axiom for deducing the axiom (12), $Reg(\lambda) \to \forall z < \lambda(\exists x, y < \lambda[z < x \wedge P(\lambda, x, y)])$, and $(\mathbf{F}^{\Sigma_1}_{x\cup\{\lambda\}})$ for proving the axiom (11), $P(\lambda, x, y) \wedge z < x \to \varphi[\lambda, z] \to \varphi^{y}[x, z]$ for $\Sigma_1$ $\varphi$. Likewise $(\mathbf{P}_I)$ and $(\mathbf{F}^{\Sigma_n}_{x})$ for the axioms (15) and (14).

Let us explain the purpose of the unusual(, though correct) inference rules $(\bigvee)$, $(\bigwedge)$ for deriving $\Sigma_n \cup \Pi_n$-formulae. For simplicity set $\lambda = \omega_1$ and $F_x = F^{\Sigma_1}_{x\cup\{\omega_1\}}$, and consider the language of ordinals. Consider the standard inference rules for introducing existential quantifiers in which any correct witness can be a witness:

$$\frac{(\mathcal{H},\kappa,n) \vdash \theta[\vec{\gamma}, \alpha], \Gamma}{(\mathcal{H},\kappa,n) \vdash \exists z < \beta\, \theta[\vec{\gamma}, z], \Gamma}$$

where $\alpha < \beta$. Then its dual should be

$$\frac{\{(\mathcal{H}[\{\alpha\}],\kappa,n) \vdash \Gamma, \neg\theta[\vec{\gamma}, \alpha]\}_{\alpha<\beta}}{(\mathcal{H},\kappa,n) \vdash \Gamma, \forall z < \beta\, \neg\theta[\vec{\gamma}, z]}$$

But then, we have to examine all possible witnesses $\alpha < \beta$ in deriving the axiom $\forall z < I \neg\theta[z, \omega_1, a], \exists z < F_x(I)\, \theta[z, F_x(\omega_1), a]$ for $a < x = F_x(\omega_1)$: Assume $a, x, y \in \mathcal{H}$.

$$\frac{\{(\mathcal{H}[\{\alpha\}],\kappa,n) \vdash \neg\theta[\alpha, \omega_1, a], \exists z < F_x(I)\, \theta[z, F_x(\omega_1), a]\}_{\alpha<I}}{(\mathcal{H},\kappa,n) \vdash \forall z < I \neg\theta[z, \omega_1, a], \exists z < F_x(I)\, \theta[z, F_x(\omega_1), a]}$$

For $\alpha \in dom(F_x)$ we can deduce it by $(\mathbf{F}_x)$

$$\cfrac{\cfrac{(\mathcal{H}[\{\alpha\}],\kappa,n) \vdash \neg\theta[\alpha, \omega_1, a], \theta[\alpha, \omega_1, a]}{(\mathcal{H}[\{\alpha\}],\kappa,n) \vdash \neg\theta[\alpha, \omega_1, a], \theta[F_x(\alpha), F_x(\omega_1), a]}\ (\mathbf{F}_x)}{(\mathcal{H}[\{\alpha\}],\kappa,n) \vdash \neg\theta[\alpha, \omega_1, a], \exists z < F_x(I)\, \theta[z, F_x(\omega_1), a]}$$

But there are ordinals $\alpha < I$ such that $\alpha \not\in dom(F_x)$ since $dom(F_x) = \mathrm{Hull}^I_{\Sigma_1}(x \cup \{\omega_1\})$ is countable, and $I > \omega_1$ is uncountable.

Moreover the same trouble occurs, when an inference rule for quantifiers followed by an $(\mathbf{F}_x)$:

$$\dfrac{\dfrac{\Gamma, \theta[\vec{\gamma}, \alpha]}{\Gamma, \exists z < \beta\, \theta[\vec{\gamma}, z]}}{\Gamma, \exists z < F_x(\beta)\, \theta[F_x(\vec{\gamma}), z]}\ (\mathbf{F}_x)$$

Even if $\alpha < \beta$, it may be the case $\alpha \not\in dom(F_x)$. Then one can not replace a cut inference with its cut formula $\exists z < F_x(\beta)\, \theta[F_x(\vec{\gamma}), z]$ by one of a cut formula of the form $\theta[F_x(\vec{\gamma}), F_x(\alpha')]$.

Contrary to this, in the inference rule for $\delta = \mu z < \beta\, \theta[\vec{\gamma}, z]$,

$$\dfrac{(\mathcal{H}, \kappa, n) \vdash \Gamma, \theta[\vec{\gamma}, \delta]}{(\mathcal{H}, \kappa, n) \vdash \Gamma, \exists z < \beta\, \theta[\vec{\gamma}, z]}\ (\bigvee)$$

$\delta$ is $\Sigma_1$-definable from $\{\beta\} \cup \vec{\gamma}$ if $\beta < I$. Therefore if $\{\beta\} \cup \vec{\gamma} \subset dom(F_x)$, then so is $\delta$.

We will state some lemmata for the operator controlled derivations with sketches of their proofs since these can be shown as in [6].

In what follows by an operator we mean an $\mathcal{H}_{\gamma,n}[\Theta]$ for a finite set $\Theta$ of sets.

**Lemma 5.12** (Tautology) *If* $\mathsf{k}(\Gamma \cup \{A\}) \subset \mathcal{H}$, *then* $(\mathcal{H}, I, n) \vdash^{2\mathrm{rk}(A)}_0 \Gamma, \neg A, A$.

**Lemma 5.13** ($\Sigma_n \cup \Pi_n$-completeness)
*For any sentence* $A \in \Sigma_n \cup \Pi_n$, $(A \text{ is true}) \Rightarrow (\mathcal{H}, I, n) \vdash^{2\mathrm{rk}(A)}_0 A$.

**Proof**. This is seen by induction on the number of symbols occurring in $\Sigma_n \cup \Pi_n$-sentences $A$. $\Box$

**Lemma 5.14** (Elimination of false $\Sigma_n$-sentences)
*For any sentence* $A \in \Sigma_n$, $(A \text{ is false})\, \&\, (\mathcal{H}, I, n) \vdash^a_c \Gamma, A \Rightarrow (\mathcal{H}, I, n) \vdash^a_c \Gamma$.

**Proof**. This is seen by induction on $a$ using Proposition 5.6. $\Box$

**Lemma 5.15** *Let* $\varphi[x, z] \in \Sigma_m$ *for* $m \geq 1$, *and* $\Theta_c = \{\neg\forall y(\forall x \in y\, \varphi[x, c] \to \varphi[y, c])\}$. *Then for any operator* $\mathcal{H}$, *and any* $a, c$, $(\mathcal{H}[\{c, a\}], I, n) \vdash^{I+2m+4+2\mathrm{rk}_L(a)}_{I+m+2} \Theta_c, \forall x \in a\, \varphi[x, c]$.

**Proof** by induction on $\mathrm{rk}_L(a)$. Let $f(a) = I + 2m + 4 + 2\mathrm{rk}_L(a)$. By IH we have for any $b \in a$, $(\mathcal{H}[\{c, b\}], I, n) \vdash^{f(b)}_{I+m+2} \Theta_c, \forall x \in b\, \varphi[x, c]$. On the other hand by Lemma 5.12 with $\mathrm{rk}(\varphi) \leq I + m - 1$ and $\mathrm{rk}(\forall x \in b\, \varphi[x, c]) \leq I + m + 1$, we have $(\mathcal{H}[\{c, b\}], I, n) \vdash^{I+2m+4}_0 \Theta_c, \neg\forall x \in b\, \varphi[x, c], \varphi[b, c]$. By a (cut) with $I + 2m + 4 \leq f(b)$ we obtain $(\mathcal{H}[\{c, b\}], I, n) \vdash^{f(b)+1}_{I+m+2} \Theta_c, \varphi[b, c]$. $(\bigwedge)$ yields $(\mathcal{H}[\{c, a\}, I, n) \vdash^{f(a)}_{I+m+2} \Theta_c, \forall x \in a\, \varphi[x, c]$. $\Box$

**Definition 5.16** $(\mathcal{H}, I, n) \vdash_c^{<\alpha} \Gamma :\Leftrightarrow \exists\beta < \alpha[(\mathcal{H}, I, n) \vdash_c^{\beta} \Gamma]$.

**Lemma 5.17** *Let $A$ be an axiom in $\mathrm{T}(I, n)$ except Foundation axiom schema. Then $(\mathcal{H}, I, n) \vdash_0^{<I+\omega} A$ for any operator $\mathcal{H} = \mathcal{H}_{\gamma,n}$.*

**Proof**. By Lemma 5.13 there remains nothing to show for $\Pi_2$-axioms in $\mathrm{KP}\omega+(V = L)$.

We consider the axiom (11). Let a $\Sigma_1$-formula $\varphi[x, a] \equiv \exists z \in L_I\, \theta[z, x, a]$ be given, and assume $\lambda, \iota, \nu, a \in L_I$.

**Case 1** : $\lambda \in R \,\&\, \iota = \Psi_{\lambda,n}\alpha$ with $\alpha \in \mathcal{H}_{\alpha,n}(\iota) \,\&\, \nu = F^{\Sigma_1}_{\iota\cup\{\lambda\}}(I)$ and $a \in L_\iota$.

We show $(\mathcal{H}[\{\lambda, \iota, a\}], I, n) \vdash_0^{<I} \neg\varphi[\lambda, a], \varphi^\nu[\iota, a]$.

Let $c = \mu z \in L_I\theta[z, \lambda, a]$. Then $\mathrm{rk}(\theta[c, \lambda, a]) < I$ since $\theta$ is $\Delta_0$, and by Lemma 5.12 we obtain $(\mathcal{H}[\{\lambda, a\}], I, n) \vdash_0^{<I} \neg\theta[c, \lambda, a], \theta[c, \lambda, a]$, where $c \in \mathrm{Hull}^I_{\Sigma_1}(\{\lambda, a\}) \subset \mathcal{H}[\{\lambda, a\}]$.

By $(\mathbf{F}^{\Sigma_1}_{\iota\cup\{\lambda\}})$ with $\iota = F^{\Sigma_1}_{\iota\cup\{\lambda\}}(\lambda)$, $a = F^{\Sigma_1}_{\iota\cup\{\lambda\}}(a)$, $(\mathcal{H}[\{\lambda, \iota, a\}], I, n) \vdash_0^{<I} \neg\theta[c, \lambda, a], \theta[F^{\Sigma_1}_{\iota\cup\{\lambda\}}(c), \iota, a]$, where $F^{\Sigma_1}_{\iota\cup\{\lambda\}}(c) \in \mathcal{H}[\{\lambda, \iota, a\}]$ by $c \in \mathcal{H}[\{\lambda, \iota, a\}]$.

By $F^{\Sigma_1}_{\iota\cup\{\lambda\}}(c) = \mu z \in L_\nu\theta[z, \iota, a] \in L_\nu$ for $\nu = F^{\Sigma_1}_{\iota\cup\{\lambda\}}(I)$, and $(\bigvee)$, $(\mathcal{H}[\{\lambda, \iota, a\}], I, n) \vdash_0^{<I} \neg\theta[c, \lambda, a], \varphi^\nu[\iota, a]$, where $\nu \in \mathcal{H}[\{\lambda, \iota, a\}]$. By $(\bigwedge)$ we conclude $(\mathcal{H}[\{\lambda, \iota, a\}], I, n) \vdash_0^{<I} \neg\varphi[\lambda, a], \varphi^\nu[\iota, a]$.

**Case 2** : Otherwise.

Then $\neg P(\lambda, \iota, \nu) \simeq \bigwedge\emptyset$ or $\neg(a \in L_\iota) \simeq \bigwedge\emptyset$. Hence $(\mathcal{H}[\{\lambda, \iota, \nu, a\}], I, n) \vdash_0^0 \neg P(\lambda, \iota, \nu), \neg(a \in L_\iota)$.

So in any case, $(\mathcal{H}[\{\lambda, \iota, \nu, a\}], I, n) \vdash_0^{<I} \neg P(\lambda, \iota, \nu), \neg(a \in L_\iota), \neg\varphi[\lambda, a], \varphi^\nu[\iota, a]$.

By $(\bigvee)$ and $(\bigwedge)$ we obtain $(\mathcal{H}, I, n) \vdash_0^I \forall\lambda, a, x, y \in L_I\{P(\lambda, x, y) \to a \in L_x \to \varphi[\lambda, a] \to \varphi^y[x, a]\}$. Note that $P(\lambda, x, y) \to a \in L_x \to \varphi[\lambda, a] \to \varphi^y[x, a]$ is not a $\Sigma_n$-formula since the predicate $P$ occurs in it.

Likewise the axiom (14) is derived by $(\mathbf{F}^{\Sigma_n}_x)$, and $(\mathcal{H}, I, n) \vdash_0^{<I+\omega}$ (14).

Finally consider the axiom (13). If $a$ is not an ordinal, then $(\mathcal{H}[\{a\}], I, n) \vdash_0^{<I} a \notin Ord$ for a $\Delta_0$-formula $Ord$. Hence $(\mathcal{H}[\{a\}], I, n) \vdash_0^{<I} a \in Ord \to \exists y[y > a \wedge Reg(y)]$. Assume $a$ is an ordinal. By Proposition 4.2.6 and $n \geq 2$ we have $a^+ \in \mathcal{H}[\{a\}]$, and $(\mathcal{H}[\{a\}], I, n) \vdash_0^{<I} a^+ > a \wedge Reg(a^+)$, and $(\mathcal{H}[\{a\}], I, n) \vdash_0^{<I} a \in Ord \to \exists y[y > a \wedge Reg(y)]$. Therefore by $(\bigwedge)$ we obtain $(\mathcal{H}, I, n) \vdash_0^I \forall x \in Ord\exists y[y > x \wedge Reg(y)]$. □

**Lemma 5.18** (Embedding)
*If $\mathrm{T}(I, n) \vdash \Gamma[\vec{x}]$, there are $m, k < \omega$ such that for any $\vec{a} \subset L_I$, $(\mathcal{H}[\vec{a}], I, n) \vdash_{I+m}^{I\cdot2+k} \Gamma[\vec{a}]$ for any operator $\mathcal{H} = \mathcal{H}_{\gamma,n}$.*

**Proof**. By Lemma 5.15 we have $(\mathcal{H}, I, n) \vdash_{I+m+2}^{I\cdot2} \forall u, z(\forall y(\forall x \in y\,\varphi[x, z] \to \varphi[y, z]) \to \varphi[u, z])$ for $\varphi[x, z] \in \Sigma_m$. By Lemmata 5.12 and 5.17 it suffices to consider inference rules of logical connectives.

Suppose $(\mathcal{H}[\{a,b\}],I,n)\vdash^{I\cdot 2+k}_{I+m}\Gamma[a,b],\theta[a]$ and $(\mathcal{H}[\{a,b\}],I,n)\vdash^{I\cdot 2+k}_{I+m}\Gamma[a,b],a\in b$ for any $a\in L_I$, where we suppress parameters for simplicity. We show for $A\equiv\exists z\in b\,\theta[z]$

$$\forall a\in L_I\{(\mathcal{H}[\{a,b\}],I,n)\vdash^{<I\cdot 2+\omega}_{I+m}\Gamma[a,b],A\} \tag{22}$$

If $\exists z\in b\,\theta[z]\not\in\Sigma_n$, then there is nothing to prove. Assume $\exists z\in b\,\theta[z]\in\Sigma_n$. If $\exists z\in b\,\theta[z]$ is true (in $L$), then by Lemma 5.13 we have $(\mathcal{H}[\{b\}],I,n)\vdash^{2\mathrm{rk}(A)}_{0}A$, and hence (22) follows.

Otherwise $(a\not\in b)\vee\neg\theta[a]$ is true. If $\theta[a]$ is false, by Lemma 5.14 we have $(\mathcal{H}[\{a,b\}],I,n)\vdash^{I\cdot 2+k}_{I+m}\Gamma[a,b]$, and hence (22). Otherwise $a\in b$ is false, by Lemma 5.14 we have $(\mathcal{H}[\{a,b\}],I,n)\vdash^{I\cdot 2+k}_{I+m}\Gamma[a,b]$, and hence (22).

Next assume that $a$ does not occur in $\Gamma$. Then wee can choose $a$ as we wish. If $\exists z\in b\,\theta[z]$ is true, then let $a=\mu z\in b\,\theta[z]\in\mathrm{Hull}^I_{\Sigma_1}(\mathsf{k}(A))$. $(\mathcal{H}[\{b\}],I,n)\vdash^{<I\cdot 2+\omega}_{I+m}\Gamma[b],\theta[a]$ yields (22) by $(\bigvee)$ and $\mathrm{rk}_L(a)<I$. Otherwise let $a=0$. $(0\not\in b)\vee\neg\theta[0]$ is true. The rest is the same as above.

The case for introducing a universal quantifier is similar to the existential case. □

**Corollary 5.19** *Assume* $\mathrm{T}(I,n)\vdash\theta$ *for a sentence* $\theta$. *Let* $m_0$ *be a number such that* $\varphi\in\Sigma_{m_0}$ *if an instance* $\forall u,z(\forall y(\forall x\in y\,\varphi[x,z]\to\varphi[y,z])\to\varphi[u,z])$ *of Foundation axiom schema occurs in the given* $\mathrm{T}(I,n)$*-proof of* $\theta$.

*Then for* $m=\max\{m_0+10,n+7\}$, $(\mathcal{H},I,n)\vdash^{<I\cdot 2+\omega}_{I+m}\varphi$ *for any operator* $\mathcal{H}=\mathcal{H}_{\gamma,n}$.

**Proof**. This is seen from the proof of Lemma 5.18, and $\mathrm{rk}(\forall u,z(\forall y(\forall x\in y\,\varphi[x,z]\to\varphi[y,z])\to\varphi[u,z]))\leq I+m_0+9$ if $\varphi\in\Sigma_{m_0}$ and $\mathrm{rk}(A)\leq I+n+6$ for the universal closure $A$ of instances of axioms (10)-(15) in $\mathrm{T}(I,n)$. Specifically for $A\equiv(\forall x,a(P_{I,n}(x)\to a\in L_x\to\varphi[a]\to\varphi^x[a]))$ of (14) with $\varphi\in\Sigma_n$, we have $\mathrm{rk}(A)\leq I+n+6$. □

**Lemma 5.20** (Inversion)
*Let* $d=\mu z\in b\,\theta[\vec{c},z]$ *for* $\theta\in\Pi_{n-1}$. *Then*

$$(\mathcal{H},\kappa,n)\vdash^a_b\Gamma,\exists z\in b\,\theta[\vec{c},z]\Rightarrow(\mathcal{H},\kappa,n)\vdash^a_b\Gamma,d\in b\wedge\theta[\vec{c},d]$$

*and*

$$(\mathcal{H},\kappa,n)\vdash^a_b\Gamma,\forall z\in b\,\neg\theta[\vec{c},z]\Rightarrow(\mathcal{H},\kappa,n)\vdash^a_b\Gamma,d\in b\to\neg\theta[\vec{c},d]$$

**Proof**. Consider the case when $\theta\in\Delta_0$ and $(\exists z\in b\,\theta[\vec{c},z])\equiv(\exists z\in F^{\Sigma_1}_{\iota\cup\{\lambda\}}(b_0)\,\theta[F^{\Sigma_1}_{\iota\cup\{\lambda\}}(\vec{c}_0),z])$ is a main formula of an $(\mathbf{F}^{\Sigma_1}_{\iota\cup\{\lambda\}})$ for an $\iota=\Psi_{\lambda,n}\alpha$.

We have $\{b_0\}\cup\vec{c}_0\subset dom(F^{\Sigma_1}_{\iota\cup\{\lambda\}})$. Then $d=\mu z\in F^{\Sigma_1}_{\iota\cup\{\lambda\}}(b_0)\,\theta[F^{\Sigma_1}_{\iota\cup\{\lambda\}}(\vec{c}_0),z]=F^{\Sigma_1}_{\iota\cup\{\lambda\}}(d_0)$ for $d_0=\mu z\in b_0\,\theta[\vec{c}_0,z]\in dom(F^{\Sigma_1}_{\iota\cup\{\lambda\}})$. Thus $d_0\in b_0\wedge\theta[\vec{c}_0,d_0]$ is a minor formula with its main $d\in b\wedge\theta[\vec{c},d]$ of the $(\mathbf{F}^{\Sigma_1}_{\iota\cup\{\lambda\}})$. □

In the following Lemma 5.21.2, note that $\mathrm{rk}(\exists x<\lambda\exists y<\lambda[\alpha<x\wedge P(\lambda,x,y)])=\lambda+1$ for $\alpha<\lambda\in R$, and $\mathrm{rk}(\exists x<I[\alpha<x\wedge P_{I,n}(x)])=I$.

**Lemma 5.21** (Reduction)
*Let* $C \simeq \bigvee(C_\iota)_{\iota\in J}$.

1. *Suppose* $C \not\in \{\exists x < \lambda \exists y < \lambda[\alpha < x \land P(\lambda, x, y)] : \alpha < \lambda \in R\} \cup \{\exists x < I[\alpha < x \land P_{I,n}(x)] : \alpha < I\}$. *Then*
$$(\mathcal{H}, \kappa, n) \vdash^a_c \Delta, \neg C \,\&\, (\mathcal{H}, \kappa, n) \vdash^b_c C, \Gamma \,\&\, \mathrm{rk}(C) \leq c \Rightarrow (\mathcal{H}, \kappa, n) \vdash^{a+b}_c \Delta, \Gamma$$

2. *Assume* $C \equiv (\exists x < \lambda \exists y < \lambda[\alpha < x \land P(\lambda, x, y)])$ *for an* $\alpha < \lambda \in R$ *and* $\beta \in \mathcal{H}_{\beta,n}$. *Then*
$$(\mathcal{H}_{\beta,n}, \kappa, n) \vdash^a_b \Gamma, \neg C \Rightarrow (\mathcal{H}_{\beta+1,n}, \kappa, n) \vdash^a_b \Gamma$$

3. *Assume* $C \equiv (\exists x < I[\alpha < x \land P_{I,n}(x)])$ *for an* $\alpha < I$ *and* $\beta \in \mathcal{H}_{\beta,n}$. *Then*
$$(\mathcal{H}_{\beta,n}, \kappa, n) \vdash^a_b \Gamma, \neg C \Rightarrow (\mathcal{H}_{\beta+1,n}, \kappa, n) \vdash^a_b \Gamma$$

**Proof**.
5.21.1 by induction on $b < \omega_{n+1}(I+1)$. Consider the case when both $C$ and $\neg C$ are main formulae. First consider the case when $C \equiv (F^{\Sigma_1}_{\iota\cup\{\lambda\}}\text{"}\varphi)$ is a main formula of an $(\mathbf{F}^{\Sigma_1}_{\iota\cup\{\lambda\}})$ with a $\varphi \in \Sigma_1$, and $\neg C$ is a main formula of a $(\bigwedge)$. Let $\neg C \equiv \neg F^{\Sigma_1}_{\iota\cup\{\lambda\}}\text{"}\varphi \equiv \forall z \in F^{\Sigma_1}_{\iota\cup\{\lambda\}}(e)\neg\theta[F^{\Sigma_1}_{\iota\cup\{\lambda\}}(\vec{e}), z]$ with $\vec{e} \subset \mathrm{Hull}^I_{\Sigma_1}((\mathcal{H} \cap \iota) \cup \{\lambda\})$, $e \in \mathrm{Hull}^I_{\Sigma_1}((\mathcal{H} \cap \iota) \cup \{\lambda\}) \cup \{I\}$ and for the set $d = \mu z \in F^{\Sigma_1}_{\iota\cup\{\lambda\}}(e)\, \theta[F^{\Sigma_1}_{\iota\cup\{\lambda\}}(\vec{e}), z]$ its minor formula is $\neg\theta[F^{\Sigma_1}_{\iota\cup\{\lambda\}}(\vec{e}), d]$.

For any $z \in \mathrm{Hull}^I_{\Sigma_1}(\iota \cup \{\lambda\})$ we have $F_{\iota,\lambda}\text{"}\theta[\vec{e}, z] \Leftrightarrow \theta[\vec{e}, z]$.

Now consider the set $d_0 = \mu z \in e\, \theta[\vec{e}, z]$. Then $d_0 \in \mathrm{Hull}^I_{\Sigma_1}(\iota \cup \{\lambda\}) = dom(F^{\Sigma_1}_{\iota\cup\{\lambda\}})$, and $F^{\Sigma_1}_{\iota\cup\{\lambda\}}(d_0) = d$. Moreover by $\{e\} \cup \vec{e} \subset \mathcal{H}$ we have $d_0 \in \mathcal{H}$.

By Lemma 5.20, inversion on the main formula $\forall z \in F^{\Sigma_1}_{\iota\cup\{\lambda\}}(e)\neg\theta[F^{\Sigma_1}_{\iota\cup\{\lambda\}}(\vec{e}), z]$ of the $(\bigwedge)$, we get $(\mathcal{H}, \kappa, n) \vdash^a_c \Delta, \neg\theta[F^{\Sigma_1}_{\iota\cup\{\lambda\}}(\vec{e}), F^{\Sigma_1}_{\iota\cup\{\lambda\}}(d_0)]$, and inversion on the minor formula $\exists z \in e\, \theta[\vec{e}, z]$ of $(\mathbf{F}^{\Sigma_1}_{\iota\cup\{\lambda\}})$ we get $(\mathcal{H}, \kappa, n) \vdash^{b_0}_c \theta[\vec{e}, d_0]$ for the $d_0 \in e$, and then by $(\mathbf{F}^{\Sigma_1}_{\iota\cup\{\lambda\}})$ go back to $\neg\theta[F^{\Sigma_1}_{\iota\cup\{\lambda\}}(\vec{e}), F^{\Sigma_1}_{\iota\cup\{\lambda\}}(d_0)]$.

Transfer
$$\dfrac{\Delta, \forall z \in F^{\Sigma_1}_{\iota\cup\{\lambda\}}(e)\neg\theta[F^{\Sigma_1}_{\iota\cup\{\lambda\}}(\vec{e}), z] \quad \dfrac{\exists z \in e\, \theta[\vec{e}, z], \Gamma, \Lambda}{\exists z \in F^{\Sigma_1}_{\iota\cup\{\lambda\}}(e)\theta[F^{\Sigma_1}_{\iota\cup\{\lambda\}}(\vec{e}), z], F^{\Sigma_1}_{\iota\cup\{\lambda\}}\text{"}\Gamma, \Lambda}\,(\mathbf{F}^{\Sigma_1}_{\iota\cup\{\lambda\}})}{\Delta, F^{\Sigma_1}_{\iota\cup\{\lambda\}}\text{"}\Gamma, \Lambda}\,(cut)$$

to
$$\dfrac{\Delta, \neg\theta[F^{\Sigma_1}_{\iota\cup\{\lambda\}}(\vec{e}), F^{\Sigma_1}_{\iota\cup\{\lambda\}}(d_0)] \quad \dfrac{\theta[\vec{e}, d_0], \Gamma, \Lambda}{\theta[F^{\Sigma_1}_{\iota\cup\{\lambda\}}(\vec{e}), F^{\Sigma_1}_{\iota\cup\{\lambda\}}(d_0)], F^{\Sigma_1}_{\iota\cup\{\lambda\}}\text{"}\Gamma, \Lambda}\,(\mathbf{F}^{\Sigma_1}_{\iota\cup\{\lambda\}})}{\Delta, F^{\Sigma_1}_{\iota\cup\{\lambda\}}\text{"}\Gamma, \Lambda}\,(cut)$$

Next consider the case $(\mathbf{F}_\iota)$ vs. $(\mathbf{F}_{\iota_1})$ with $\iota_1 > \iota$, where $F_\iota = F^{\Sigma_1}_{\iota,\lambda}$ for some $\lambda \in R$ or $F_\iota = F^{\Sigma_n}_\iota$ with $\lambda = I$, and similarly for $F_{\iota_1}$.

Let $F_\iota"\varphi$ be a main formula of $(\mathbf{F}_\iota)$, and $\neg F_\iota"\varphi \equiv \neg F_{\iota_1}"\theta$ a main formula of $(\mathbf{F}_{\iota_1})$.

Then by $\iota_1 > \iota$ and Proposition 4.3 we have $F_\iota(I) < \iota_1$, and hence $F_{\iota_1}"F_\iota"\varphi \equiv F_\iota"\varphi \equiv F_{\iota_1}"\theta$, i.e., $\theta \equiv F_\iota"\varphi$.

$$\dfrac{\dfrac{\Lambda, \Gamma, \varphi}{\Lambda, F_\iota"\Gamma, F_\iota"\varphi}\,(\mathbf{F}_\iota) \quad \dfrac{\neg F_\iota"\varphi, \Lambda_1, \Gamma_1}{\neg F_\iota"\varphi, \Lambda_1, F_{\iota_1}"\Gamma_1}\,(\mathbf{F}_{\iota_1})}{\Lambda, F_\iota"\Gamma, \Lambda_1, F_{\iota_1}"\Gamma}\,(cut)$$

5.21.2. Suppose $C \equiv (\exists x < \lambda \exists y < \lambda[\alpha < x \wedge P(\lambda, x, y)])$. We have $(\mathcal{H}_{\beta,n}, \kappa, n) \vdash^a_b \Gamma, \neg\exists x < \lambda\exists y < \lambda[\alpha < x \wedge P(\lambda, x, y)]$ with $\alpha < \lambda$.

Let $\iota = \Psi_{\lambda,n}\beta$ and $\nu = F^{\Sigma_1}_{\iota\cup\{\lambda\}}(I)$. Since $\alpha \in \mathcal{H}_{\beta,n} \cap \lambda$, we have $\alpha < \Psi_{\lambda,n}\beta = \iota$. Moreover by $\beta \in \mathcal{H}_{\beta,n}$ we have $\iota, \nu \in \mathcal{H}_{\beta+1,n}$. By inversion $(\mathcal{H}_{\beta+1,n}, \kappa, n) \vdash^a_b \Gamma, \neg[\alpha < \iota \wedge P(\lambda, \iota, \nu)]$ and once again by inversion with $\neg P(\lambda, \iota, \nu) \simeq (\lambda \in \lambda)$ we have $(\mathcal{H}_{\beta+1,n}, \kappa, n) \vdash^a_b \Gamma, \alpha \not< \iota, \lambda \in \lambda$. By eliminating the false sentences $\alpha \not< \iota, \lambda \in \lambda$ we have $(\mathcal{H}_{\beta+1,n}, \kappa, n) \vdash^a_b \Gamma$.

5.21.3. This is seen as in Lemma 5.21.2 by introducing the ordinal $\iota = \Psi_{I,n}\beta$. □

**Lemma 5.22** (Predicative Cut-elimination)

1. $(\mathcal{H}, \kappa, n) \vdash^b_{c+\omega^a} \Gamma \,\&\, [c, c+\omega^a[ \cap (\{\lambda + 1 : \lambda \in R\} \cup \{I\}) = \emptyset \,\&\, a \in \mathcal{H} \Rightarrow (\mathcal{H}, \kappa, n) \vdash^{\varphi ab}_c \Gamma$.
2. *For* $\lambda \in R$, *if* $\omega^b < \omega_{n+1}(I + 1)$, $(\mathcal{H}_{\gamma,n}, \kappa, n) \vdash^b_{\lambda+2} \Gamma \,\&\, \gamma \in \mathcal{H}_{\gamma,n} \Rightarrow (\mathcal{H}_{\gamma+b,n}, \kappa, n) \vdash^{\omega^b}_{\lambda+1} \Gamma$.
3. *If* $\omega^b < \omega_{n+1}(I + 1)$, $(\mathcal{H}_{\gamma,n}, \kappa, n) \vdash^b_{I+1} \Gamma \,\&\, \gamma \in \mathcal{H}_{\gamma,n} \Rightarrow (\mathcal{H}_{\gamma+b,n}, \kappa, n) \vdash^{\omega^b}_I \Gamma$.
4. $(\mathcal{H}_{\gamma,n}, \kappa, n) \vdash^b_{c+\omega^a} \Gamma \,\&\, [c, c+\omega^a[ \cap R^+ = \emptyset \,\&\, a \in \mathcal{H}_{\gamma,n} \Rightarrow (\mathcal{H}_{\gamma+\varphi ab,n}, \kappa, n) \vdash^{\varphi ab}_c \Gamma$.

**Proof**. 5.22.4. This follows from Lemmata 5.22.1, 5.22.2 and 5.22.3 using the facts $\varphi ab \geq b$, and $a > 0 \Rightarrow \varphi 0(\varphi ab) = \varphi ab$. □

**Definition 5.23** For a formula $\exists x \in d\, A(x)$ and ordinals $\lambda = \mathrm{rk}_L(d) \in R^+, \alpha$, $(\exists x \in d\, A)^{(\exists\lambda|\alpha)}$ denotes the result of restricting the *outermost existential quantifier* $\exists x \in d$ to $\exists x \in L_\alpha$, $(\exists x \in d\, A)^{(\exists\lambda|\alpha)} \equiv (\exists x \in L_\alpha\, A)$.

In what follows $F_{x,\lambda}$ denotes $F^{\Sigma_1}_{x,\lambda}$ when $\lambda \in R$, and $F^{\Sigma_n}_x$ when $\lambda = I$.

**Lemma 5.24** (Boundedness) *Let* $\lambda \in R^+$, $C \equiv (\exists x \in d\, A)$ *and* $C \notin \{\exists x < \lambda\exists y < \lambda[\alpha < x \wedge P(\lambda, x, y)] : \alpha < \lambda \in R\} \cup \{\exists x < I[\alpha < x \wedge P_{I,n}(x)] : \alpha < I\}$. *Assume that* $\mathrm{rk}(C) = \lambda = \mathrm{rk}_L(d)$.

1. $(\mathcal{H}, \lambda, n) \vdash^a_c \Lambda, C \,\&\, a \leq b \in \mathcal{H} \cap \lambda \Rightarrow (\mathcal{H}, \lambda, n) \vdash^a_c \Lambda, C^{(\exists\lambda|b)}$.
2. $(\mathcal{H}, \kappa, n) \vdash^a_c \Lambda, \neg C \,\&\, b \in \mathcal{H} \cap \lambda \Rightarrow (\mathcal{H}, \kappa, n) \vdash^a_c \Lambda, \neg(C^{(\exists\lambda|b)})$.

**Proof** by induction on $a < \omega_{n+1}(I+1)$.

Note that if a main formula $F_{\iota,\sigma}"\varphi$ of an $(\mathbf{F}_{\iota,\sigma})$ is in $\Sigma^{\Sigma_{n+1}}(\lambda)$, then either $\sigma \leq \lambda$ and there occurs no bounded quantifier $Qx < \lambda$ in $F_{\iota,\sigma}"\varphi$, or $\sigma > \iota > \lambda$ and $(F_{\iota,\sigma}"\varphi)^{(\exists\lambda|b)} \equiv F_{\iota,\sigma}"\varphi$.

Let $C \simeq \bigvee(C_\iota)_{\iota\in J}$ for $C_\iota \equiv A(\iota)$, and $(\mathcal{H},\lambda,n) \vdash_c^{a(\iota)} \Lambda, C, C_\iota$ with an $a(\iota) < a$ for an $\iota \in J = d$. Otherwise $C^{(\exists\lambda|b)} \equiv C$ by the definition. Then $C^{(\exists\lambda|b)} \simeq \bigvee(C\iota)_{\iota\in J'}$ where $J' = L_b$. By the condition (21) we have $\mathrm{rk}_L(\iota) < \lambda \Rightarrow \mathrm{rk}_L(\iota) < a \leq b$, and hence $\iota \in L_b = J'$. By IH we have Lemmata 5.24.1 and 5.24.2. $\Box$

**Lemma 5.25** (Collapsing)
*Let $\lambda \in R^+$ and $\sigma \in R^+ \cup \{\omega_\alpha : \text{limit } \alpha < I\}$.*

*Suppose $\{\gamma,\lambda,\sigma\} \subset \mathcal{H}_{\gamma,n}[\Theta]$ with $\forall\rho \geq \lambda[\Theta \subset \mathcal{H}_{\gamma,n}(\Psi_{\rho,n}\gamma)]$, and $\Gamma \subset \Sigma^{\Sigma_{n+1}}(\lambda)$. Let $\mu = \sigma+1$ if $\sigma \in R^+$. Otherwise $\mu = \sigma$ if $\sigma = \omega_\alpha$ for a limit $\alpha < I$. Then for $\hat{a} = \gamma + \omega^{\sigma+a}$ and $\kappa = \max\{\sigma,\lambda\}$, if $\hat{a} < \omega_{n+1}(I+1)$,*

$$(\mathcal{H}_{\gamma,n}[\Theta],\kappa,n) \vdash_\mu^a \Gamma \Rightarrow (\mathcal{H}_{\hat{a}+1,n}[\Theta],\lambda,n) \vdash_{\Psi_{\lambda,n}\hat{a}}^{\Psi_{\lambda,n}\hat{a}} \Gamma.$$

**Proof** by main induction on $\mu$ with subsidiary induction on $a$.

First note that $\Psi_{\lambda,n}\hat{a} \in \mathcal{H}_{\hat{a}+1,n}[\Theta] = \mathcal{H}_{\hat{a}+1,n}(\Theta)$ since $\hat{a} = \gamma + \omega^{\sigma+a} \in \mathcal{H}_{\gamma,n}[\Theta] \subset \mathcal{H}_{\hat{a}+1,n}[\Theta]$ by the assumption, $\{\gamma,\lambda,\sigma,a\} \subset \mathcal{H}_{\gamma,n}[\Theta]$.

Assume $(\mathcal{H}_{\gamma,n}[\Theta][\Lambda],\kappa,n) \vdash_\mu^{a_0} \Gamma_0$ with $\forall\rho \geq \lambda[\Lambda \subset \mathcal{H}_{\gamma,n}(\Psi_{\rho,n}\gamma)]$. Then by $\gamma \leq \hat{a}$, we have for any $\rho \geq \lambda$, $\hat{a_0} \in \mathcal{H}_{\gamma,n}[\Theta][\Lambda] \subset \mathcal{H}_{\gamma,n}(\Psi_{\rho,n}\gamma) \subset \mathcal{H}_{\hat{a},n}(\Psi_{\rho,n}\hat{a})$. This yields that

$$a_0 < a \Rightarrow \forall\rho \geq \lambda(\Psi_{\rho,n}\widehat{a_0} < \Psi_{\rho,n}\hat{a}) \tag{23}$$

Second observe that $\mathsf{k}(\Gamma) \subset \mathcal{H}_{\gamma,n}[\Theta] \subset \mathcal{H}_{\hat{a}+1,n}[\Theta]$ by $\gamma \leq \hat{a}+1$.

Third we have

$$\forall\rho \geq \lambda[\mathsf{k}(\Gamma) \subset \mathcal{H}_{\gamma,n}(\Psi_{\rho,n}\gamma)] \tag{24}$$

**Case 1**. First consider the case: $\Gamma \ni A \simeq \bigwedge\{A_\iota : \iota \in J\}$

$$\frac{\{(\mathcal{H}_{\gamma,n}[\Theta \cup \{\iota\}],\kappa,n) \vdash_\mu^{a(\iota)} \Gamma, A_\iota : \iota \in J\}}{(\mathcal{H}_{\gamma,n}[\Theta],\kappa,n) \vdash_\mu^a \Gamma}\ (\bigwedge)$$

where $a(\iota) < a$ for any $\iota \in J$. We claim that

$$\forall\iota \in J\forall\rho \geq \lambda(\iota \in \mathcal{H}_{\gamma,n}(\Psi_{\rho,n}\gamma)) \tag{25}$$

Consider the case when $A \equiv \forall x \in b\,\neg A'$. There are two cases to consider. First consider the case when $J = \{d\}$ for the set $d = \mu x \in b\, A'$. Then $\iota = d = (\mu x \in b\, A') \in \mathrm{Hull}^I_{\Sigma_n}(\mathsf{k}(A)) \subset \mathcal{H}_{\gamma,n}(\Psi_{\rho,n}\gamma)$ by (24).

Otherwise $\mathrm{rk}_L(b) < \lambda$, i.e., $b \in L_\lambda$. Let $\rho \geq \lambda$. We have $b \in \mathsf{k}(A) \subset \mathcal{H}_{\gamma,n}[\Theta] \subset \mathcal{H}_{\gamma,n}(\Psi_{\rho,n}\gamma)$. Hence $b \in \mathcal{H}_{\gamma,n}(\Psi_{\rho,n}\gamma) \cap L_\rho$. Since $\mathcal{H}_{\gamma,n}(\Psi_{\rho,n}\gamma) \cap \rho \subset \Psi_{\rho,n}\gamma$ and $\Psi_{\rho,n}\gamma$ is a multiplicative number, we have $\mathcal{H}_{\gamma,n}(L_{\Psi_{\rho,n}\gamma}) \cap L_\rho = \mathcal{H}_{\gamma,n}(\Psi_{\rho,n}\gamma) \cap L_\rho \subset L_{\Psi_{\rho,n}\gamma}$. Therefore $\iota \in b \in L_{\Psi_{\rho,n}\gamma} \subset \mathcal{H}_{\gamma,n}(\Psi_{\rho,n}\gamma)$ as desired.

Hence (25) was shown.

SIH yields

$$\frac{\{(\mathcal{H}_{\widehat{a(\iota)}+1,n}[\Theta\cup\{\iota\}],\lambda,n)\vdash^{\Psi_{\lambda,n}\widehat{a(\iota)}}_{\Psi_{\lambda,n}\widehat{a(\iota)}}\Gamma,A_\iota:\iota\in J\}}{(\mathcal{H}_{\hat{a}+1,n}[\Theta],\lambda,n)\vdash^{\Psi_{\lambda,n}\hat{a}}_{\Psi_{\lambda,n}\hat{a}}\Gamma}\ (\bigwedge)$$

for $\widehat{a(\iota)}=\gamma+\omega^{\sigma+a(\iota)}$, since $\Psi_{\lambda,n}\widehat{a(\iota)}<\Psi_{\lambda,n}\hat{a}$ by (23).
**Case 2**. Next consider the case for an $A\simeq\bigvee\{A_\iota:\iota\in J\}\in\Gamma$ and an $\iota\in J$ with $a(\iota)<a$ and $\mathrm{rk}_L(\iota)<\kappa\Rightarrow\mathrm{rk}_L(\iota)<a$

$$\frac{(\mathcal{H}_{\gamma,n}[\Theta],\kappa,n)\vdash^{a(\iota)}_{\mu}\Gamma,A_\iota}{(\mathcal{H}_{\gamma,n}[\Theta],\kappa,n)\vdash^{a}_{\mu}\Gamma}\ (\bigvee)$$

Assume $\mathrm{rk}_L(\iota)<\lambda$. We show $\mathrm{rk}_L(\iota)<\Psi_{\lambda,n}\hat{a}$. By $\Psi_{\lambda,n}\gamma\leq\Psi_{\lambda,n}\hat{a}$, it suffices to show $\mathrm{rk}_L(\iota)<\Psi_{\lambda,n}\gamma$.

Consider the case when $A\equiv\exists x\in b\,A'$. There are two cases to consider. First consider the case when $J=\{d\}$ for the set $d=\mu x\in b\,A'$. Then $\iota=d=(\mu x\in b\,A')\in\mathrm{Hull}^{I}_{\Sigma_n}(\mathsf{k}(A))$, and $\mathrm{rk}_L(\iota)\in\mathrm{Hull}^{I}_{\Sigma_n}(\mathsf{k}(A))\subset\mathcal{H}_{\gamma,n}(\Psi_{\lambda,n}\gamma)$ by (24). If $\mathrm{rk}_L(\iota)<\lambda$, then $\mathrm{rk}_L(\iota)\in\mathcal{H}_{\gamma,n}(\Psi_{\lambda,n}\gamma)\cap\lambda\subset\Psi_{\lambda,n}\gamma$.

Otherwise we have $J=b\in\mathsf{k}(A)\subset\mathcal{H}_{\gamma,n}[\Theta]$, and we can assume that $\iota\in\mathsf{k}(A_\iota)\subset\mathcal{H}_{\gamma,n}[\Theta]$. Otherwise set $\iota=0$. We have $\mathrm{rk}_L(\iota)<\mathrm{rk}_L(b)\leq\lambda$, and $\mathrm{rk}_L(\iota)\in\mathcal{H}_{\gamma,n}(\Psi_{\lambda,n}\gamma)\cap\lambda\subset\Psi_{\lambda,n}\gamma$.

SIH yields for $\widehat{a(\iota)}=\gamma+\omega^{\sigma+a(\iota)}$

$$\frac{(\mathcal{H}_{\widehat{a(\iota)}+1,n}[\Theta],\lambda,n)\vdash^{\Psi_{\lambda,n}\widehat{a(\iota)}}_{\Psi_{\lambda,n}\widehat{a(\iota)}}\Gamma,A_\iota}{(\mathcal{H}_{\hat{a}+1,n}[\Theta],\lambda,n)\vdash^{\Psi_{\lambda,n}\hat{a}}_{\Psi_{\lambda,n}\hat{a}}}\ (\bigvee)$$

**Case 3**. Third consider the case for an $a_0<a$ and a $C$ with $\mathrm{rk}(C)<\mu$.

$$\frac{(\mathcal{H}_{\gamma,n}[\Theta],\kappa,n)\vdash^{a_0}_{\mu}\Gamma,\neg C\quad(\mathcal{H}_{\gamma,n}[\Theta],\kappa,n)\vdash^{a_0}_{\mu}C,\Gamma}{(\mathcal{H}_{\gamma,n}[\Theta],\kappa,n)\vdash^{a}_{\mu}\Gamma}\ (cut)$$

**Case 3.1**. $\mathrm{rk}(C)<\lambda$.

We have by (24) $\mathsf{k}(C)\subset\mathcal{H}_{\gamma,n}(\Psi_{\lambda,n}\gamma)$. Proposition 5.9.5 yields $\mathrm{rk}(C)\in\mathcal{H}_{\gamma,n}(\Psi_{\lambda,n}\gamma)\cap\lambda\subset\Psi_{\lambda,n}\gamma\leq\Psi_{\lambda,n}\hat{a}$. By Proposition 5.9.3 we see that $\{\neg C,C\}\subset\Sigma^{\Sigma_{n+1}}(\lambda)$.

SIH yields for $\widehat{a_0}=\gamma+\omega^{\sigma+a_0}$

$$\frac{(\mathcal{H}_{\widehat{a_0}+1,n}[\Theta],\lambda,n)\vdash^{\Psi_{\lambda,n}\widehat{a_0}}_{\Psi_{\lambda,n}\widehat{a_0}}\Gamma,\neg C\quad(\mathcal{H}_{\widehat{a_0}+1,n}[\Theta],\lambda,n)\vdash^{\Psi_{\lambda,n}\widehat{a_0}}_{\Psi_{\lambda,n}\widehat{a_0}}C,\Gamma}{(\mathcal{H}_{\hat{a}+1,n}[\Theta],\lambda,n)\vdash^{\Psi_{\lambda,n}\hat{a}}_{\Psi_{\lambda,n}\hat{a}}\Gamma}\ (cut)$$

**Case 3.2**. $\lambda\leq\mathrm{rk}(C)<\mu$ and $\mathrm{rk}(C)\not\in R^+$.

Let $\pi:=\min\{\pi\in R^+:\pi>\mathrm{rk}(C)\}$. We have $\pi\in R$ and $\pi\in\mathcal{H}_{\gamma,n}[\Theta]$ by $\mathrm{rk}(C)\in\mathcal{H}_{\gamma,n}[\Theta]$ and Proposition 4.2.6.

Then $\lambda \leq \text{rk}(C) < \pi < \mu$, and hence $\{\neg C, C\} \subset \Sigma^{\Sigma_{n+1}}(\pi)$ by Proposition 5.9.3. SIH with $\max\{\pi, \sigma\} = \sigma = \kappa$ yields for $\widehat{a_0} = \gamma + \omega^{\sigma + a_0}$ and $\beta = \Psi_{\pi,n}\widehat{a_0}$, $(\mathcal{H}_{\widehat{a_0}+1,n}[\Theta], \pi, n) \vdash^{\beta}_{\beta} \Gamma, \neg C$ and $(\mathcal{H}_{\widehat{a_0}+1,n}[\Theta], \pi, n) \vdash^{\beta}_{\beta} C, \Gamma$.

Let $\mu' = \omega_\alpha + 1 < \beta$ for $\pi = \omega_{\alpha+1}$. Then $\beta = \mu' + \omega^\beta$ and $[\mu', \mu' + \omega^\beta[ \cap R^+ = \emptyset$. Moreover $\text{rk}(C) < \beta$. By a $(cut)$

$$\frac{(\mathcal{H}_{\widehat{a_0}+1,n}[\Theta], \pi, n) \vdash^{\beta}_{\mu'+\omega^\beta} \Gamma, \neg C \quad (\mathcal{H}_{\widehat{a_0}+1,n}[\Theta], \pi, n) \vdash^{\beta}_{\mu'+\omega^\beta} C, \Gamma}{(\mathcal{H}_{\widehat{a_0}+1,n}[\Theta], \pi, n) \vdash^{\beta+1}_{\mu'+\omega^\beta} \Gamma}\ (cut)$$

Predicative Cut-elimination 5.22 yields

$$(\mathcal{H}_{\widehat{a_0}+\varphi\beta(\beta+1),n}[\Theta], \pi, n) \vdash^{\varphi\beta(\beta+1)}_{\mu'} \Gamma$$

We have $\mu' < \mu$. MIH with $\max\{\lambda, \mu'\} < \pi$ and Proposition 5.11 yields

$$(\mathcal{H}_{\widehat{a_1}+1,n}[\Theta], \lambda, n) \vdash^{\Psi_{\lambda,n}\widehat{a_1}}_{\Psi_{\lambda,n}\widehat{a_1}} \Gamma$$

for $\widehat{a_1} = \widehat{a_0} + \varphi\beta(\beta+1) + \omega^{\omega_\alpha + \varphi\beta(\beta+1)} = \gamma + \omega^{\sigma+a_0} + \omega^{\omega_\alpha+\varphi\beta(\beta+1)} < \gamma + \omega^{\sigma+a} = \hat{a}$ by $a_0 < a$, $\omega_\alpha < \sigma$ and $\beta < \sigma$ with a strongly critical $\sigma$. Thus $\Psi_{\lambda,n}\widehat{a_1} < \Psi_{\lambda,n}\hat{a}$ and $(\mathcal{H}_{\hat{a}+1,n}[\Theta], \lambda, n) \vdash^{\Psi_{\lambda,n}\hat{a}}_{\Psi_{\lambda,n}\hat{a}} \Gamma$.

**Case 3.3**. $\lambda \leq \text{rk}(C) < \mu$ and $\pi := \text{rk}(C) \in R^+$.

Then $C \in \Sigma^{\Sigma_{n+1}}(\pi)$ and $\pi \leq \sigma$. Also $\pi \in \mathcal{H}_{\gamma,n}[\Theta]$. $C$ is either a sentence $\exists x < I[\alpha < x \land P_{I,n}(x)]$ with $\pi = I$, or a sentence $\exists x \in d\, A(x)$ with $\mathsf{qk}(A) < \pi = \text{rk}_L(d) \leq I$.

In the first case we have $\kappa = \sigma = I$, and $(\mathcal{H}_{\gamma+1,n}[\Theta], I, n) \vdash^{a_0}_{I+1} \Gamma$ by Reduction 5.21.3, and IH yields the lemma.

Consider the second case. From the right uppersequent, SIH with $\max\{\pi, \sigma\} = \sigma = \kappa$ yields for $\widehat{a_0} = \gamma + \omega^{\sigma+a_0}$ and $\beta_0 = \Psi_{\pi,n}\widehat{a_0} \in \mathcal{H}_{\widehat{a_0}+1,n}[\Theta]$

$$(\mathcal{H}_{\widehat{a_0}+1,n}[\Theta], \pi, n) \vdash^{\beta_0}_{\beta_0} C, \Gamma$$

Then by Boundedness 5.24.1 and $\beta_0 \in \mathcal{H}_{\widehat{a_0}+1,n}[\Theta]$, we have

$$(\mathcal{H}_{\widehat{a_0}+1,n}[\Theta], \pi, n) \vdash^{\beta_0}_{\beta_0} C^{(\exists\pi|\beta_0)}, \Gamma$$

On the other hand we have by Boundedness 5.24.2 from the left uppersequent

$$(\mathcal{H}_{\widehat{a_0}+1,n}[\Theta], \pi, n) \vdash^{a_0}_{\mu} \Gamma, \neg(C^{(\exists\pi|\beta_0)})$$

Moreover we have $\neg(C^{(\exists\pi|\beta_0)}) \in \Sigma^{\Sigma_{n+1}}(\pi)$. SIH yields for $\widehat{a_0} < \widehat{a_1} = \widehat{a_0} + 1 + \omega^{\sigma+a_0} = \gamma + \omega^{\sigma+a_0} + 1 + \omega^{\sigma+a_0} < \gamma + \omega^{\sigma+a} = \hat{a}$ and $\beta_1 = \Psi_{\pi,n}\widehat{a_1}$

$$(\mathcal{H}_{\widehat{a_1}+1,n}[\Theta], \pi, n) \vdash^{\beta_1}_{\beta_1} \Gamma, \neg C^{(\exists\pi|\beta_0)}$$

Now we have $\widehat{a_i} \in \mathcal{H}_{\widehat{a_i},n}(\Psi_{\pi,n}\hat{a})$ and $\widehat{a_i} < \hat{a}$ for $i < 2$, and hence $\beta_0 = \Psi_{\pi,n}\widehat{a_0} < \beta_1 = \Psi_{\pi,n}\widehat{a_1} < \Psi_{\pi,n}\hat{a}$. Therefore $\text{rk}(C^{(\exists\pi|\beta_0)}) < \beta_1 < \Psi_{\pi,n}\hat{a}$.

Consequently

$$\frac{(\mathcal{H}_{\widehat{a_1}+1,n}[\Theta],\pi,n)\vdash^{\beta_1}_{\beta_1}\Gamma,\neg C^{(\exists\pi|\beta_0)}\quad(\mathcal{H}_{\widehat{a_0}+1,n}[\Theta],\pi,n)\vdash^{\beta_0}_{\beta_0}C^{(\exists\pi|\beta_0)},\Gamma}{(\mathcal{H}_{\widehat{a_1}+1,n}[\Theta],\pi,n)\vdash^{\beta_1+1}_{\beta_1}\Gamma}\ (cut)$$

Let $(\alpha,\mu',\beta_2)=(\alpha,\omega_\alpha+1,\beta_1)$ if $\pi=\omega_{\alpha+1}$, and $(\alpha,\mu',\beta_2)=(\beta_1,\beta_1,0)=(\beta_1,\omega_{\beta_1},0)$ if $\pi=I$. Then $\beta_1\leq\mu'+\omega^{\beta_2}$ and $[\mu',\mu'+\omega^{\beta_2}[\cap R^+=\emptyset$.

Predicative Cut-elimination 5.22 yields

$$(\mathcal{H}_{\widehat{a_1}+\varphi\beta_2(\beta_1+1),n}[\Theta],\pi,n)\vdash^{\varphi\beta_2(\beta_1+1)}_{\mu'}\Gamma$$

We have $\mu'<\mu$. MIH with $\max\{\lambda,\mu'\}\leq\pi$ yields

$$(\mathcal{H}_{\widehat{a_2}+1,n}[\Theta],\lambda,n)\vdash^{\Psi_{\lambda,n}\widehat{a_2}}_{\Psi_{\lambda,n}\widehat{a_2}}\Gamma$$

for $\widehat{a_2}=\widehat{a_1}+\varphi\beta_2(\beta_1+1)+\omega^{\omega_\alpha+\varphi\beta_2(\beta_1+1)}=\gamma+\omega^{\sigma+a_0}+\omega^{\sigma+a_0}+\omega^{\omega_\alpha+\varphi\beta_2(\beta_1+1)}<\gamma+\omega^{\sigma+a}=\hat{a}$ by $a_0<a$, $\omega_\alpha<\sigma$ and $\beta_1<\sigma$ with a strongly critical $\sigma$. Thus $\Psi_{\lambda,n}\widehat{a_2}<\Psi_{\lambda,n}\hat{a}$ and $(\mathcal{H}_{\hat{a}+1,n},\lambda,n)\vdash^{\Psi_{\lambda,n}\hat{a}}_{\Psi_{\lambda,n}\hat{a}}\Gamma$.

**Case 4**. Fourth consider the case for an $a_0<a$

$$\frac{(\mathcal{H}_{\gamma,n}[\Theta],\kappa,n)\vdash^{a_0}_{\mu}\Lambda,\Gamma_0}{(\mathcal{H}_{\gamma,n}[\Theta],\kappa,n)\vdash^{a}_{\mu}\Gamma}\ (\mathbf{F})$$

where $\Gamma=\Lambda\cup F"\Gamma_0$ and either $F=F^{\Sigma_1}_{x\cup\{\rho\}}$, $\Gamma_0\subset\Sigma_1$ for some $x$ and $\rho$, or $F=F^{\Sigma_n}_x$, $\Gamma_0\subset\Sigma_n$ for an $x$. Then $\Lambda\cup\Gamma_0\subset\Sigma^{\Sigma_{n+1}}(\lambda)$. SIH yields the lemma. □

**Corollary 5.26** *Suppose $\Gamma\subset\Sigma^{\Sigma_{n+1}}(\omega_1)$. Assume $(\mathcal{H}_{0,n},I,n)\vdash^{I\cdot 2+k}_{I+m}\Gamma$ for some $m,k<\omega$ such that $b=\omega_m(I\cdot 3+k)<\omega_{n+1}(I+1)$. Let $\beta=\Psi_{\omega_1,n}(b)$ and $c=\varphi\beta\beta$. Then $(\mathcal{H}_{b+1,n},\omega_1,n)\vdash^{c}_{0}\Gamma$.*

**Proof**. Let $(\mathcal{H}_{0,n},I,n)\vdash^{I\cdot 2+k}_{I+m}\Gamma$. By Predicative Cut-elimination 5.22.3 we have $(\mathcal{H}_{0,n},I,n)\vdash^{\omega_{m-1}(I\cdot 2+k)}_{I+1}\Gamma$. Collapsing 5.25 yields $(\mathcal{H}_{b+1,n},\omega_1,n)\vdash^{\beta}_{\beta}\Gamma$. By Predicative Cut-elimination 5.22.1 we obtain $(\mathcal{H}_{b+1,n},\omega_1,n)\vdash^{c}_{0}\Gamma$. □

**Proposition 5.27** *For each sentence $A$ in the language $\{\in\}\cup L_I$ the following holds.*

1. *$A\simeq\bigvee(A_\iota)_{\iota\in J}\Rightarrow\forall\iota\in J(A_\iota$ is an $\{\in\}\cup L_I$-sentence), and similarly for the case $A\simeq\bigwedge(A_\iota)_{\iota\in J}$.*
2. *$A\simeq\bigvee(A_\iota)_{\iota\in J}\Leftrightarrow(L_I\models A\Leftrightarrow\exists\iota\in J(L_I\models A_\iota))$.*
3. *$A\simeq\bigwedge(A_\iota)_{\iota\in J}\Leftrightarrow(L_I\models A\Leftrightarrow\forall\iota\in J(L_I\models A_\iota))$.*
4. *$(\mathcal{H},\omega_1,n)\vdash^{\alpha}_{0}\Gamma\ \&\ \alpha<\omega_{n+1}(I+1)\Rightarrow L_I\models\bigvee\Gamma$.*

**Proof**. Propositions 5.27.1-5.27.3 are straightforward.

Proposition 5.27.4 is proved by induction on $\alpha<\omega_{n+1}(I+1)$ using Propositions 5.27.1-5.27.3 and the fact that $(\mathbf{F}^{\Sigma_1}_{x\cup\{\lambda\}})$ and $(\mathbf{F}^{\Sigma_n}_{x})$ are truth-preserving, that is to say if the upper sequent of these inferences is true, then so is the lower sequent, cf. Proposition 5.6. □

## 6 Proof of Theorem 1.1

For a sentence $\exists x \in L_{\omega_1}\varphi$ in the language $\{\in, \omega_1\}$, assume $\mathsf{ZF} + (V = L) \vdash \exists x \in L_{\omega_1}\varphi$. Let $n_0 \geq 2$ be the number such that in the given $\mathsf{ZF} + (V = L)$-proof instances of axiom schemata of Separation and Collection are $\Sigma_{n_0}$-Separation and $\Sigma_{n_0}$-Collection, and let $n_1$ the number such that in the given $\mathsf{ZF} + (V = L)$-proof instances of Foundation axiom schema are applied to $\Sigma_{n_1}$-formulae. Let $m = \max\{n_0 + 7, n_1 + 10\}$, and let $n = m + 1$. Then by Lemma 3.2 and Corollary 5.19 we see that the fact $(\mathcal{H}_{0,n}, I, n) \vdash^{<I\cdot 2+\omega}_{I+m} \exists x \in L_{\omega_1}\varphi$ is provable in $\mathsf{ZF} + (V = L)$. We have $b = \omega_m(I \cdot 3 + \omega) < \omega_n(I + 1)$. In what follows work in $\mathsf{ZF} + (V = L)$. Corollary 5.26 yields $(\mathcal{H}_{b+1,n}, \omega_1, n) \vdash^c_0 \exists x \in L_{\omega_1}\varphi$ for $\beta = \Psi_{\omega_1,n}(b)$ and $c = \varphi\beta\beta$. Boundedness 5.24 yields $(\mathcal{H}_{b+1,n}, \omega_1, n) \vdash^c_0 \exists x \in L_c\,\varphi$. Then by Proposition 5.27.4 with $c < \Psi_{\omega_1,n}\omega_n(I + 1)$ we obtain $\exists x \in L_{\Psi_{\omega_1,n}\omega_n(I+1)}\varphi$.

The whole proof is formalizable in $\mathsf{ZF} + (V = L)$, we conclude $\mathsf{ZF} + (V = L) \vdash \exists x \in L_{\Psi_{\omega_1,n}\omega_n(I+1)}\varphi$. This completes a proof of Theorem 1.1.

**Remark**. Using notation systems of infinitary derivations as in [7], it is reasonable to expect the following:

Over a weak base theory T, $\mathsf{ZF} + (V = L)$ is a conservative extension of $\mathrm{T} + (V = L) + \{\exists x < \omega_1[x = \Psi_{\omega_1,n}\omega_n(I + 1)] : n < \omega\}$ with respect to a class of formulae depending on T.

Since any cut-free derivation of a first-order sentence is finite in depth, we actually have the following Corollary 6.1.

**Corollary 6.1** *Assume* $\mathsf{ZF} + (V = L) \vdash \exists x < \omega\,\varphi$. *Then there exist* $n, h < \omega$ *such that*

$$(\mathcal{H}_{\omega_n(I+1)+1,n}, \omega_1, n) \vdash^h_0 \exists x < \omega\,\varphi.$$

**Problem**. Let $g$ be the Gödel number of a $\mathrm{T}(I)$-proof of $\exists x < \omega\,\varphi$, and $h = H(g)$ a bound of depth of cut-free derivation. Note here that a number $n < \omega$ such that $(\mathcal{H}_{\omega_n(I+1)+1,n}, \omega_1, n) \vdash^h_0 \exists x < \omega\,\varphi$ is calculable from $g$. Then the map $H$ on $\omega$ seems not to be provably total in $\mathsf{ZF} + (V = L)$, i.e., $\mathsf{ZF} + (V = L) \not\vdash \forall g \in \omega \exists h \in \omega[h = H(g)]$, and $H \not\in L_{\Psi_{\omega_1}\varepsilon_{I+1}}$.

The problem is to find a reasonable hierarchy of reals$\in {}^{\omega}\omega$ indexed by countable ordinals, and to show that $H$ is too rapidly growing to be provably total in $\mathsf{ZF} + (V = L)$.

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